Riyaziyyat və Mexanikanın Müasir Problemləri

Ümummilli Lider Heydər Əliyevin 100-illik
yubileyinə həsr olunmuş Beynəlxalq Konfransın

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Modern Problems of Mathematics and Mechanics

PROCEEDINGS

of the International Conference dedicated to the
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Biography of Heydar Aliyev

Heydar Alirza oglu Aliyev was born on May 10, 1923 in the city of Nakhchivan, Azerbaijan, in the family of a simple railway worker. Although he is considered to be from Nakhchivan, his parents - Alirza and Izzet are from West Azerbaijan. Their roots begin with two brothers named Jomard and Alagoz Mammad. According to the researchers of this lineage, the brothers were born in Ardabil and held high positions in the army of the Iranian ruler Nadir Shah Afshar (1736-1747). In 1738, Panahali Khan, one of Nadir Shah's most respected and influential generals, fled to Karabakh after a conflict with Javanshir Shah. After the death of Nadir Shah, Panahali Khan founded the independent Karabakh Khanate. The center of the khanate was first Shahbulag fortress, and since 1751 it was rebuilt on the ruins of the ancient city of Shusha and named Panahabad in honor of the person who founded it. Jomard and Alagoz went to the side of Muhammad Panahali Khan, won brilliant military victories under his leadership and united a number of Azerbaijani lands to the Karabakh Khanate. At that time (in the middle of the XVIII century), Zangezur district was part of the Karabakh Khanate. After completing their military service, Panahali Khan gave them a plot of land on the southeastern slopes of the Zangezur Mountains. In the 60s of the XVIII century, Jomard and Mohammad built two villages named Jomardli and Mohammadli in that area. Alirza's grandfather Jafar was a wealthy man. This is proved by the fact that he went to Karbala twice. At that time, it was not every man's job to go to Karbala. In 1905-1906, after the Armenian bandits raided the villages of Zangezur district, robbed the villagers, burned their houses, and looted their livestock, the family of Karbalayi Jafar living in Jomardli went bankrupt. Alirza was married to a common man named Naringul in his early manhood, his first son Hasan was born in 1907, and Huseyn was born in 1911. He is forced to leave his family members with his parents and go to Baku in search of income. In 1918, the genocidal policy of the Armenians against the Turks reached Zangezur. As a result of this bloody incident, Alirza's two brothers, his brothers' wives, children and others were brutally killed. The only surviving brother, Zeynalabdin, took the family of Alirza, his wife Naringul with a nursing baby in her arms, and his sons Hasan and Huseyn out of the seage and moved to Nakhchivan. While passing through the mountain passes, the baby dies. Soon, Naringul could not bear these losses and died. Alirza was not aware of these events. After two years, he came to Nakhchivan, and after a long search, he was able to find his brother. In 1922, Alirza married Izzet, a distant relative. They rent a one-room apartment in the house of Najaf from Nakhchivan and start living. Izzat did not separate Alirza's sons Hasan and Huseyn from her daughter Shura born from her previous husband, and accepts them as her own children.

After Alirza found a permanent job in Nakhchivan, he built a house on the plot of land given to him because he was a railway worker and settled there. On May 10, 1923, the first child of this family was born. He was named Heydar in honor of his uncle Heydar, who died in the Jomardli genocide. Then Aqil, Jalal, Shafiga and Rafiga were born. When Heydar reached school age, his parents put him in the Azerbaijani section of Nakhchivan city international school number one. After finishing the seventh grade, in 1936 he decided to enter pedagogical school in Nakhchivan. However, he could study in international school until the tenth grade. It seems that his desire to become a teacher was related to the great prestige of teaching at that time. In the summer of 1939, Heydar Aliyev graduated from the Nakhchivan City Pedagogical School with an honors diploma, receiving excellent grades in all 30 exams. After graduating from the Nakhchivan Pedagogical Technical College in 1939, Heydar Aliyev went to Baku to continue his education. His dream was to become an architect, to take part in the construction of cities by developing the art he loved since childhood, and to study architecture, which is the art of construction. For this purpose, he enters the Faculty of Architecture of the Azerbaijan Industrial Institute (now Azerbaijan State Oil and Industry University). However, this dream does not come true. Since living in Baku was very difficult financially, Heydar Aliyev had to leave his studies at the end of 1940 and returned to Nakhchivan due to the financial situation of his younger brothers and sisters living in Nakhchivan. In 1941-1943, he worked in various responsible positions in the People's Commissariat of Internal Affairs of the Nakhchivan MSSR and the Council of People's Commissars of the Nakhchivan ASSR, and at the end of 1943, he was sent to the state security agency by the Nakhchivan Provincial Party Committee. This message was not from his heart. Firstly, because he was always inclined towards a civil profession, he wanted to become an architect, and he intended to return to the Azerbaijan Industrial Institute as soon as the war ended. The second, by moving to the State Security Committee, lost a lot in both salary and food ration. In 1943, his father, Alirza, died, and the livelihood of a large family fell on him.

Thirdly, while working in the Commissariat of Internal Affairs and the Council of People's Commissars, Heydar Aliyev was closely acquainted with the methods used by the Soviet secret police in the 1930s and 1940s, and he did not like these methods at all. Despite all this, he could not refuse the offer to go to the authorities. Time has shown that the comrades who recommended him to this field were not wrong in their choice.On January 27, 1944, 21-year-old Heydar Aliyev wrote an application to the People's Commissariat for State Security (PCSC) to be employed as an operational commissioner. In May 1944, he was already an employee of the PCSC with the rank of lieutenant. Near the end of the Second World War, the interests of the Kremlin coincided with the most lofty dreams of the Azerbaijani people, who were divided between two states, but never lost hope of reunification. The sending of Soviet troops into South Azerbaijan, the active activity of political workers and intellectuals from Soviet Azerbaijan gave impetus to democratic processes there, supported the growth of the national liberation movement, and ultimately led to the establishment of the National Government of South Azerbaijan, led by Peshawari. The implementation of all these works was included in the sphere of activity of the state security agencies of Azerbaijan, and Lieutenant Aliyev participated in security measures during the movements of Soviet and foreign government delegations to Iran and back. For example, in November 1944, when General de Gaulle was on the road in Baku, young Aliyev was busy with his protection. On April 7, 1946, the leader of Azerbaijan, Mir Jafar Bagirov, came to Julfa to meet with Peshawari, and in June and October, the meetings took place in Nakhchivan. Aliyev, one of the organizers of both meetings, personally met Mir Jafar Bagirov during these meetings. After World War II, there were thousands of prisoners who returned to the USSR, including Azerbaijan. The most unfortunate of these people were those who were captured by the Germans. The captives were considered traitors and were executed. According to various estimates, about 1.5 million people from the peoples of the USSR fought on the German side. In July 1946, more than 3 thousand captives returned to Azerbaijan. An inspection-research commission was established in all parts of Azerbaijan, including in Nakhchivan. Heydar Aliyev was also a member of that commission. He approached these inspections very carefully, and it was with his participation that hundreds of Azerbaijanis were able to be released. In 1947, Senior Lieutenant Heydar Aliyev was appointed as the head of the 5th Department of the Ministry of State Security of the Republic of Nakhchivan.

This unit was one of the most important units of the Nakhchivan security agencies, conducting counter-intelligence work against Iranian and Turkish special services. As soon as the war ended, in 1945, Heydar Aliyev was trained in short-term courses under the Special State Service Committee of Azerbaijan on increasing the Chekist training of operatives. He received excellent grades in all subjects taught here. Eight years of working in state security agencies in Nakhchivan gave him a lot both as a specialist and as a person. He proved to his compatriots, and first of all to himself, that even by working in such serious and cruel, merciless penal institutions, it is possible to remain a real person, to look straight into people's eyes without hesitation or shame. At that time, dozens of people were saved from death camps and exiles, despite the strict official rules, thanks to his intervention, and were returned to a comfortable life in their native land.

In May 1949, on the eve of his 26th birthday, news came from Baku that Heydar Aliyev was being sent to Leningrad (now St. Petersburg) to study at the school for the retraining of operative management staff of the Ministry of State Security of the USSR. In the certificate given to him upon his graduation from this school, it is said that during his studies at the school, Senior Lieutenant Heydar Aliyev showed himself only in a positive way, he took lessons seriously, he was an excellent student during his studies, he was repeatedly thanked by the head of the school, and his name was written on the honor board of the honors pupil. It is also noted in the character reference that senior lieutenant Aliyev skillfully reveals the main essence in solving operational issues, acts freely and correctly in operational conditions, makes the right decisions, compiles operational documents in a complete and justified manner, and deserves to be promoted in the official position. In 1950, Heydar Aliyev was transferred to the central apparatus of the Ministry of State Security of Azerbaijan and was appointed as the deputy head of the 6th division of the 2nd counter-intelligence department. This department was engaged in conducting counter-intelligence operations "in the eastern direction", that is, against Turkey and Iran. It is known that in the early 1940s and late 1950s, Turkey and Iran came under the full influence of the West, especially the United States, which was declared the number one enemy of the Soviet Union. In 1951, Heydar Aliyev entered the part-time department of the Faculty of History of the Azerbaijan State University, and in 1957 he graduated from that faculty with an honors diploma. When he started working in Baku, he bought a house there and took his mother and younger sister with him. About two years later, he was given a two-room apartment by the state on the old Yerevan Street (next to the current school No. 6), and the whole family gathered in this apartment. In those years, Heydar Aliyev was awarded the highest awards of the Center for the preparation and implementation of a number of successful special operations in the direction of Iran and Turkey. As a result, in 1953, he was appointed the head of the second department of the Ministry of State Security for Baku region. 7 years after moving to Baku, his mother Izzet died of heart failure in October 1957 at the age of 62. In 1958, 35-year-old Heydar Aliyev headed the counter-intelligence department of Azerbaijan. On March 31, 1965, Heydar Aliyev was given the rank of colonel. In January 1965, Heydar Aliyev was appointed to the position of chairman of the State Security Committee of Azerbaijan. On May 19, 1967, Yuri Vladimirovich Andropov was appointed to the position of chairman of the State Security Committee of the USSR. On June 22, 1967, Heydar Aliyev was appointed chairman of the State Security Committee of Azerbaijan and worked in this position until July 14, 1969.

Heydar Aliyev worked in two directions in the State Security Committee of Azerbaijan. The first is to Azerbaijanize the system, and the second is to protect intellectuals from persecution. In the 1950s and 1960s, by recruiting zealous and national-minded sons of the Motherland to work in the State Security Committee, he actually prevented further persecutions against the honest sons of our nation. On the other hand, it was not everyone's skill to save the Azerbaijani intellectuals who had national ideas in those difficult years - Academician Ziya Bunyadov, Khalil Rza Uluturk, Bakhtiyar Vahabzade, Anar and others - from persecution for their national ideas, risking their civil position, their lives, and their future. However, Heydar Aliyev not only protected them from danger, but also prepared a fertile ground for the further development of their scientific and artistic creativity, further strengthening of their positions, and created all kinds of conditions. Speaking about the lack of political dissidents among creative intellectuals in the republic during the years when he led Azerbaijan, the Great Leader said: "...If we were to look for them, we could have found many dissidents." But we weren't looking. What does it mean? This means that we heartily accepted the works of those dissidents and created opportunities for them to find their way. Thus, we expressed our solidarity with those authors."

On July 14, 1969, Heydar Aliyev was elected as the first secretary of the Central Committee of the Communist Party of Azerbaijan, a leader of Azerbaijan, one of the most backward republics in the USSR. At that time, the leadership of the USSR needed to create a rule of law in Azerbaijan that would satisfy Moscow, neutralize the forces stirring up national interests, and destroy the seeds of nationalism. In Moscow, they openly said that Heydar Aliyev, the head of the State Security Committee, knows the weak points very well and can soon create the law and order we want in Azerbaijan. As a result of the wise and purposeful policy implemented by the Great Leader, Azerbaijan, which was known as an agrarian province of the USSR until 1969, ranked 14th among the 15 allied republics, after that date became a developed industrial country, a republic where scientific and technical progress was widely applied. could become a country known all over the world for its high culture. In this not simple situation, despite the pressure from Moscow, various kinds of protests of the dismissed personnel, all kinds of sabotages, written anonymous letters, provocations and other obstacles, the locomotive had already set off. He would serve his people honorably until 2003, in all forms and methods, until the end of his life...

As one of the people who lived at that time and was lucky enough to witness those great events, I well remember that Heydar Aliyev's speech in Azerbaijani at the 50th anniversary of Baku State University in November 1969 was the second major political event after the August plenum. First of all, I would like to note that in those years, Baku State University, which maintained its status as a flagship university by acting as the main center of national-progressive thought and public thought, and took the main burden in national personnel training, and strengthening the activities of this educational institution, was a special request of the great leader. was in the spotlight. I want to emphasize one point that Heydar Aliyev was so diverse and wide-ranging that those working in the oil field thought that he was only interested in the oil issue, those working in agriculture thought that he was only interested in this field, in transport, art, education, those working in other areas of the industry also thought that he was only engaged in that area. We were sure that since he graduated from Baku State University, he pays special attention to this university. Much later, we saw that he pays special attention and care to other higher schools of the republic, and to the education system in general, as an important field that forms the intellectual base of the country. It was the result of this concern that immediately after coming to power, he headed the university's 50th anniversary commission. The great leader's participation and speech at this solemn event on November 1, 1969 at the club named after Dzerzhinsky (now Shahriyar club) was welcomed by the university staff with great satisfaction. Let's note one fact that the last time the jubilee of BSU was held in 1929. In the following years, the celebration of such significant events was not so important. Only after 40 years - in 1969, the celebration of the 50th anniversary at the initiative of the National leader was considered a great and significant stage in the glorious history of BSU.

It was a historical event that Heydar Aliyev spoke in Azerbaijani at the jubilee ceremony, overcoming all the barriers imposed by the totalitarian Soviet regime. High representatives from the leadership of the Central Committee of the CPSU, the Minister of Higher and Secondary Education of the USSR, ministers from Moscow, St. Petersburg, Kiev and other cities, rectors and other Russian-speaking guests participated in that event. It should be noted that even though no one was officially forbidden to speak in their native language at that time, high-ranking persons usually gave their speeches only in Russian at events of this kind until Heydar Aliyev. Despite these bans, Heydar Aliyev spoke in Azerbaijani and showed everyone that everyone can speak in his own language, and if we want to achieve the development of our mother language, we should go in this way.

I would like to bring to your attention some of the projects implemented in our country during Heydar Aliyev's leadership of Azerbaijan in 1969-1982 and of great importance in the life of our people:

1. Commissioning of Baku household air conditioners plant, "Azon" Science and Production Union, Electronic calculating machines plant, deep foundations plant, Azerbaijan SDPS, Sumgait compressor plant, Top knitting factory, Baku sewing and shoe factories, Ganja, non-ferrous metals processing plant, Nakhchivan glassware plant, Ali Bayramli household appliances plant, which play an important role in the economic life of the country.

2. Launching of tea factories in Astara and Lankaran, champagne wine factory in Baku, "Istisu" mineral water plant in Kalbajar, spinning mill in Sumgait, wool primary processing factories in Yevlakh;

3. For the purpose of increasing oil production, the creation of new floating facilities for drilling wells with a depth of 6 kilometers in the fields of the Caspian Sea at a depth of 70 meters.

4. Increasing grape production from 272 thousand tons in 1969 to 2 million tons in 1983;

5. Increasing cotton production three times in 12 years to one million tons;

6. Putting Tertarchay, Arpachay, Agstafachay, Khanbulanachay, etc. water basins into use and, as a result, increasing the irrigated land areas to 1 million 450 thousand hectares;

7. Construction and commissioning of up to 30 large poultry complexes based on new technology, including Siyazan, Davachi, Yeni Baku broiler, Absheron breeding poultry factories:

8. 2.5 times increase of national income in the Republic;

9. Commissioning of hundreds of kilometers of new railways and highways, hundreds of bridges, river crossings in our country, metro stations "Ulduz", "Azizbayov", "Gara Garayev", "Neftchilar", "Nizami" in Baku city;

10. Construction and commissioning of 18 million 293 thousand square meters of residential area in the Republic, construction of a number of residential areas, microdistricts of Baku, Ganja, Sumgayit, housing conditions of more than 1 million 700 thousand people were improved, increasing the number of Azerbaijanis living in Baku, from 33% to 70%;

11. The construction and commissioning of Giant hotels such as "Absheron", "Moscow", "Azerbaijan", "Tabriz", the Republic Palace, the current Presidential Palace, the Presidential Residence, the building of the National Assembly, as well as the "Gulustan" Palace, the Sports-Concert Complex, the Sports Palace of Hand Games, and dozens of other buildings, sports and cultural complexes;

12. From 1969 to the mid-1980s, construction of about 700 school buildings that meet modern needs in cities, villages and towns of Azerbaijan, opening of five new higher schools, increasing the number of higher school students from 70,000 to 100,000;

13. From the 1970s to 1987, more than 15,000 young Azerbaijanis were sent to study abroad;

14. Starting from 1972, every year 300-400 young Azerbaijanis are sent to various higher schools of the USSR to receive higher military education;

15. Awarding the Hero of Socialist Labor, the highest award of the USSR, to such poets, writers, artists, teachers as Suleyman Rustam, Rasul Rza, Suleyman Rahimov, Mirza Ibrahimov, Gara Garayev, Fikret Amirov, Rashid Behbudov, Mahar Guliyev.

His activity was highly appreciated by the leadership of the USSR. In March 1976, he was elected a member of the Political Bureau of the Central Committee of the Communist Party of the Soviet Union, and in August 1979, he was awarded the honorary title of Hero of Socialist Labor. On November 22, 1982, he was given higher trust, was elected a member of the Political Bureau of the Central Committee of the Communist Party of the Soviet Union, and was appointed to the post of First Deputy Chairman of the Council of Ministers of the USSR. With this, a new period in his life and activity began. The period of activities such as leading the socio-economic and cultural development of the entire USSR, directing the economic development strategy of the Soviet government, implementing the foreign policy of the USSR as a prominent politician and statesman, and fulfilling the responsible and difficult tasks for the establishment of mutual relations with foreign countries has begun. . By the decision of the Presidium of the Council of Ministers of the USSR dated December 15, 1982, to Heydar Alirza oglu Aliyev, the First Deputy Chairman of the Council of Ministers was assigned to a) Supervise the USSR Ministry of Roads, the Ministry of the Navy, the Ministry of Road Construction, the Ministry of Communications, as well as the issues of automobile, river transport and road infrastructure of the allied republics; b) to lead the permanent commission of the Soviet of Ministers of the USSR on issues of Baikal-Amur railway construction. At the meeting of the Presidium of the Council of Ministers of the USSR held on February 2, 1983, Heydar Aliyev, the first deputy chairman of the Council of Ministers, was entrusted to lead the All-Union Ministry of Health, the Ministry of Medical Industry, the Ministry of Culture, the Ministry of Education, the Ministry of Higher and Secondary Education, the State Technical and Vocational Education Committee, the State Broadcasting Committee, the State Cinematography Committee, the State Publishing House, the Sports Committee, the Soviet Union Telegraph Agency, the State Archive and Copyright Agency. Heydar Aliyev highly valued the transport system of the Soviet Union, which was located in a large area, and its decisive role in the development of the economy. He considered transport to be the blood circulation of the entire country. He led the work of transport with his own concept. He believed that transport management should be approached from an economic, political and psychological point of view. By the way, witnesses remember that during the years when he was in charge of the transport of the USSR, which had a huge railway network from Moldova to Eastern Siberia, during the selector meetings held once a week (at that time there were no other electronic means), Heydar Aliyev not only knew all the heads of hundreds of junctions by name, he also made such decisive and precise decisions that no one expected that the experts working in this field for decades remained speechless. In 1983, Heydar Aliyev was awarded the honorary title of Hero of Socialist Labor for the second time. In 1984, he led the commissions created in connection with the commissioning of the 770-kilometer Baikal-Amur highway, the crashed steam locomotive "Suvorov" and the passenger ship "Admiral Nakhimov", speeding up work on the 15-kilometer-long Northern Muysk tunnel, and the entire transport system of the USSR, education , science, culture, heavy industry, trade, light industry, health, Middle East and Latin American politics, communications, emergencies, agriculture, construction and dozens of other complex areas were under his authority. At that time, the great son of our people was also a powerful obstacle, an insurmountable barrier in front of the Armenian nationalists who tried to take Nagorno-Karabakh away from Azerbaijan. That is why Gorbachev, who was a submissive tool in the hands of the Armenian mafia from Stavropol, was finally able to achieve his seditious intention: in October 1987, the genius son of our nation was removed from the leadership of the USSR. In fact, this was the heaviest blow inflicted on our people during all the years of Soviet rule. It is no coincidence that immediately after that, the Armenian nationalists supported by Gorbachev started an active struggle to separate Nagorno-Karabakh from Azerbaijan. The scenario of the events that happened after that was written in advance by Gorbachev and his Armenian patrons: the deportation of our compatriots from Armenia, the atrocities in Sumgait, the occupation of the Karabakh lands, the January 20 tragedy, the Ganja uprising and what else...

When our country was in an ungovernable situation, everyone already knew that Heydar Aliyev should return to Azerbaijan. However, since the Moscow government knew this well, they kept him under strict control and persecution by all means, and did not allow him to leave the sanatorium where he lived. The State Security Committee, which was acting for 30 years under the instructions of Gorbachev, also pursued him as a person against the Soviet system, and they wanted to force him to eliminate him through provocation and stop the struggle for truth, justice, and the Motherland. But nothing could silence Heydar Aliyev. Immediately after the events of January 20, he came to the permanent representation of Azerbaijan in Moscow, raised his protest voice, held a press conference, named the culprits, and declared that he was with the people. Despite all prohibitions, on July 20, 1990, he secretly arrived at the airport and flew to Baku and from there to Nakhchivan. In the same year, he was elected a deputy to the Nakhchivan Supreme Council, and on September 3, 1991, he was elected the chairman of the Supreme Council of the Nakhchivan Autonomous Republic. On November 21, 1992, Heydar Aliyev was elected the chairman of the new party at the founding conference of the New Azerbaijan Party held in Nakhchivan. In 1991-1993, the Great Leader was able to preserve its territorial integrity as a result of the wise policy. In May-June 1993, when there was a threat of civil war and loss of independence in the country, the people of Azerbaijan stood up with the demand to bring Heydar Aliyev to power. The then leaders of Azerbaijan were forced to officially invite Heydar Aliyev to Baku. On June 15, 1993, he was elected the chairman of the Supreme Soviet of Azerbaijan, and on July 24, by the decision of the Milli Majlis, he began to exercise the powers of the President of the Republic of Azerbaijan. On October 3, 1993, as a result of the national vote, Heydar Aliyev was elected the President of the Republic of Azerbaijan. On October 11, 1998, he was re-elected as the President of the Republic of Azerbaijan, having collected 76.1 percent of the votes in the elections held in conditions of high public participation. As a result of Heydar Aliyev's return to power, in a short period of time, our country was saved from the chaos, danger of civil war, and disintegration in 1988-92. Thanks to his determination to be a statesman, the uprising attempt and protest actions started in Ganja under the leadership of Colonel Surat Huseynov in June 1993, and the existence of the separatist "Talish-Mugan Republic" started in Lankaran under the leadership of Alikram Humbatov in September of the same year, also ended thanks to the courage and determination of the Great Leader. The infamous Sadwal separatist movement in the north was crushed and thus the fragmentation of our country into several parts was prevented in a bloodless and peaceful way and the country was saved from the threat of civil war. In October 1994, the attempted coup d'état under the leadership of then Prime Minister Surat Huseynov, as well as the insurrection against statehood launched by brothers Rovshan and Mahir Javadov on March 13-17, 1995, were prevented thanks to his courage, determination and wise management experience. With this, the infamous intentions of the internal and external forces who wanted to divide Azerbaijan into different parts, end its independence, disrupt social and political stability in our country, and seize power, were thwarted. The Great Leader saved our independence by relying on the patriotic and zealous representatives of the people: simple people, young people, intellectuals. The great leader was re-elected as the President of the Republic of Azerbaijan on October 11, 1998, having collected 76.1 percent of the votes in the elections held in conditions of high public participation. I would like to bring to your attention some development-oriented events carried out under the leadership of the Great Leader in 1993-2003:

1. First of all, successful steps were taken in the direction of restoring security, rule of law, and preservation of independence. Attempts to seize power by armed means were stopped once and for all. Firm, lasting stability was ensured in the republic;

2. The deviations of the Armenian armed forces were prevented, strong counter-blows were delivered to the enemy, and thus, in May 1994, a ceasefire was declared on the front line as the first step in the resolution of the conflict. There was an end to the senseless bloodshed and the failure of the army's manpower. Illegal armed groups, which were the cause of many arbitrariness in the country, were gradually abolished, and the formation of a regular army began;

3. Azerbaijan's foreign policy activity was developed in many directions, Heydar Aliyev's international relations with the USA, Russia, Great Britain, France, Germany, Poland, Romania, Japan, People's Republic of China, Turkey, Iran, Pakistan, etc. as a result of his official visits to the leading countries, the establishment of equal relations between our countries was achieved, our relations with Turkey reached a qualitatively new stage, based on the "one nation, two states" formula, a decision was made on an unshakable foundation;

4. In September 1994, the signing of the large oil contract, which was called the "Contract of the Century", and later, the signing of new contracts related to the exploitation of hydrocarbon resources of the Caspian sector of Azerbaijan, creating opportunities for the revival of the economic potential of our country, the large-scale flow of billions of dollars of foreign capital, which the economy needs today, to the republic;

5. In September 1998, holding the first international conference on the restoration of the Historical Silk Road with the participation of the high representatives of 32 countries and 14 international organizations, including the presidents of 7 countries;

6. Admission of Azerbaijan to the full membership of the Council of Europe in January 2001;

On December 12, the national leader of Azerbaijan, President Heydar Aliyev died in the Cleveland Clinic of the United States of America, and was buried on December 15 in Baku, in the Alley of Honor. Dear and brilliant memory of Heydar Aliyev, who left deep and indelible traces in the history of Azerbaijan, will never be erased from the hearts of our appreciative people. Azerbaijan, which founded by him, will live forever.

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Influence of sanding on the dynamics of gas movement

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Sanding during the operation of oil and gas wells has a negative impact on oil and gas production. In this case, the sand carried out by the fluid flow accumulates in the bottom-hole zone, clogs the passage of the pipe string, and thereby increases the hydraulic resistance. This reduces the influx of fluid per unit of time from the reservoir to the well and in some cases, in strongly sandy wells can lead to its complete failure [1, 2].

In this paper, we consider the movement of gas in the pipe section with a permeable wall and clogged with sand, then in the pipe section clogged with sand and in its free section. For each section, we write the gas movement equation, initial and boundary conditions, and then from the continuity equation, we determine pressures at the junction of the contact of each section and then determine well-productive capacity per unit of time.

The gas filtration problem was solved on the material balance.

Gas mass G_0 in the stratum can be determined by the formula

$$G_0 = 2\pi \, m \, h \int_{r_c}^{R_k} \rho \cdot r dr, \tag{1}$$

$$\rho = \frac{P}{P_{\text{atm}}} \rho_{\text{atm}}$$

The initial and boundary conditions

$$P|_{t=0} = P_{\text{k}}, \quad r = R_k, \tag{2}$$

$$P|_{r=r_c} = P_c(t), \; t > 0, \quad \left. \frac{\partial P}{\partial r} \right|_{r=R_k} = 0, \quad t > 0, \tag{3}$$

Pressure P at any point of the stratum will be sought in the form

$$P = P_c\,(t) + A\,(t)\,f\,(r)\,, \tag{4}$$

where $A\,(t)$ is an unknown function dependent on t .

We find the function $f\left(r\right)$ satisfying the boundary conditions by the formula

$$f\left(r\right)=\ln\frac{r}{r_c}-\frac{r}{R_k}+\frac{r_c}{R_k} \tag{5}$$

Having substituted the expressions (4) and (5) in expression (1), we obtain

$$G_0=2\pi\,m\,h\frac{\rho_{atm}}{P_{atm}}\left[P_c\left(t\right)\frac{R_k^2-r_c^2}{2}+\frac{R_k^2}{2}D\,A\left(t\right)\right] \tag{6}$$

where $D=\ln\frac{R_k}{r_c}-\frac{7}{6}+\frac{1}{2}\left(\frac{r_c}{R_k}\right)^2+\frac{r_c}{R_k}-\frac{1}{3}\left(\frac{r_c}{R_k}\right)^3$. The mass flow of gas to the well per unit of time can be determined by the formula

$$G=\frac{dG_0}{dt} \tag{7}$$

$$G=-k\frac{\left(P_c(0)+P_c(T)\right)}{\mu\,\beta}\pi\,r_c h\left.\frac{\partial P}{\partial r}\right|_{r=r_c} \tag{8}$$

where $\beta=\frac{P_{\mathrm{atm}}}{\rho_{\mathrm{atm}}}$. Substituting expressions (4) and (8) in formula (8), the expression (6) in equation (7), and equating them we obtain a differential equation with respect to $A\left(t\right)$

$$\dot{A}(t)+\alpha\,A(t)=-\frac{1}{D}\dot{P}_c\left(t\right) \tag{9}$$

where $\alpha=k\frac{\left(P_c(0)+P_c(T)\right)}{\mu\,m\,R_k^2\,D}$. Having solved the differential equation (7) with regard to the initial condition (2) and substituting the obtained expression in formulas (4) and (8), we find mass gas flow.
For the second section, we will find the motion equation with regard to the permeability of the pipe wall in the form

$$\frac{\partial P}{\partial t}-\frac{c^2k\rho}{\mu}\frac{\partial^2 P}{\partial x^2}-c^2\alpha_0 P=c^2\alpha_0 P_c(t) \tag{10}$$

Initial and boundary conditions

$$\left.P\right|_{t=0}=f(x),\ \left.P\right|_{x=0}=P_c(t),\ \left.P\right|_{x=h}=P_h(t), \tag{11}$$

For the third section, the gas motion in the pipe space clogged with sand is described by the equation

$$\gamma \frac{\partial^2 P}{\partial x^2} = \frac{\partial P}{\partial r}, t > 0 \tag{12}$$

where $\gamma = \rho_q \frac{k}{\mu\beta^*}$, $\rho_q = \frac{P}{P_{\text{atm}}}\rho_{\text{atm}}$

Initial and boundary conditions

$$P|_{t=0} = \psi(x),\, P|_{x=0} = P_h(t),\, P|_{x=h_1} = P_0(t), \tag{13}$$

The equation of motion and continuity of gas in sand-free pipe space will be of the form

$$-\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial t} + 2a_1 Q, \quad -\frac{\partial P}{\partial t} = c^2 \frac{\partial Q}{\partial x} \tag{14}$$

Initial and boundary conditions have the form

$$P|_{t=0} = \phi(x),\, P|_{x=0} = P_0(t),\; P|_{x=h_2} = P_y(t), \tag{15}$$

Having solved equations (10), (12) and (14) with regard to the initial and boundary conditions (11), (13) and the continuity condition, (15) we obtain the pressure field at each section of gas motion and then its mass flow.

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The effect of time and temperature on the rheological behavior and flow performance of fluid

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The changing of the liquid rheological properties depending on temperature influences the dynamics of its movement. Although this problem has been known for a long-time, its physical origin remains poorly known. The present paper aims at giving further insight on the process of movement of a homogeneous liquid through a pipeline of a complex profile (Fig. 1).

Let us accept the following assumptions:

1) the velocity of liquid particles along the axis of the pipeline is averaged across the cross-section;

2) when determining the temperature field of the liquid along the axis of the pipeline, the perturbed part of it is neglected;

3) it is assumed that the rate of change in the kinematic viscosity of the fluid depending on the coordinates is much faster than depending on time.

The movement equation and the continuity equation, taking into account the convective term, have the form

$$-\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial t}+v_0\frac{\partial Q}{\partial x}+2aQ;\quad -\frac{1}{c^2}\frac{\partial P}{\partial t}=\frac{\partial Q}{\partial x}+\frac{v_0}{c^2}\frac{\partial P}{\partial x}. \tag{1}$$

Consider the stationary case. From expression (1) we get

$$\frac{dQ}{dx}-\frac{2\,a\,v_0}{c^2-v_0^2}Q=0. \tag{2}$$

Integrating expressions (2) and, in the first approximation, leaving only the constant term $a\,(x)$, we obtain

$$Q=C\,\exp\left(\frac{2\,a_0 v_0}{c^2-v_0^2}x\right), \tag{3}$$

C is the integration constant which is determined from the boundary condition

$$Q|_{x=0}=Q_0. \tag{4}$$

Then from expression (3) under the constrain (4), we have

$$Q=Q_0\exp\left(\frac{2\,a_0 v_0}{c^2-v_0^2}x\right). \tag{5}$$

With practical values of the system parameters, even with $x = l \ \frac{2a_0 v_0}{c^2 - v_0^2} l << 1$.

Therefore, expanding expressions (5) in a Taylor series, taking into account only the first term, we get

$$Q = Q_0 \left(1 + \frac{2\, a_0 v_0}{c^2 - v_0^2}\right) x. \tag{6}$$

The stationary thermal conductivity equation of a fluid under the accepted assumptions, taking into account only the convective term and expression (5), after some transformations, will have the form

$$\frac{dT}{dx} - T\left(1 - a_1 x^2 - b_1 x + c_1\right) = \frac{\alpha_0 \rho}{Q_0}\left(1 - \frac{2\, a_0 v_0}{c^2 - v_0^2}\right) T_1. \tag{7}$$

We have a boundary condition

$$P|_{x=0} = P_{\text{inlet}}. \tag{8}$$

Let us assume that the temperature field T_1 occurs according to a linear law (Fig. 1)

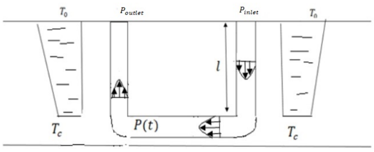

Fig.1.

$$T_1 = T_c + (T_0 - T_c)\left(1 - \frac{x}{l}\right). \tag{9}$$

Integrating expressions (7), taking into account the boundary condition (8) and expression (9), we get

$$T = \left(1 - \frac{1}{3} a_1 x^3 - \frac{1}{2} b_1 x^2 - c_1 x\right)\left(\frac{1}{6} A\, x^6 + \frac{1}{5} B\, x^5 + \frac{1}{4} C\, x^4 + \frac{1}{3} D\, x^3 + \frac{1}{2} E\, x^2 + F\, x\right) +$$

$$+ T_{\text{inlet}}\left(1 - \frac{1}{3} a_1 x^3 - \frac{1}{2} b_1 x^2 - c_1 x\right), \tag{10}$$

where

$$A = -T_0 \frac{\alpha_0 \rho}{Q_0} \frac{2\, a\, v_0}{c^2 - v_0^2} \frac{a_1}{3l}, \quad B = T_0 \frac{\alpha_0 \rho}{Q_0} \frac{2\, a\, v_0}{c^2 - v_0^2} \frac{a_1}{3} - T_0 \frac{\alpha_0 \rho}{Q_0} \frac{a_1}{3l} - T_0 \frac{\alpha_0 \rho}{Q_0} \frac{2\, a\, v_0}{c^2 - v_0^2} \frac{b_1}{2l},$$

$$C = T_0 \frac{\alpha_0 \rho}{Q_0} \frac{2\, a\, v_0}{c^2 - v_0^2} \frac{b_1}{2} - T_0 \frac{\alpha_0 \rho}{Q_0} \frac{b_1}{2l} - T_0 \frac{\alpha_0 \rho}{Q_0} \frac{2\, a\, v_0}{c^2 - v_0^2} \frac{c_1}{l} + T_0 \frac{\alpha_0 \rho}{Q_0} \frac{a_1}{3},$$

$$D=-T_0\frac{\alpha_0\rho}{Q_0}\frac{2\,a\,v_0}{c^2-v_0^2}\frac{1}{l}+T_0\frac{\alpha_0\rho}{Q_0}\frac{2\,a\,v_0}{c^2-v_0^2}\,c_1-T_0\frac{\alpha_0\rho}{Q_0}\frac{c_1}{l}+T_0\frac{\alpha_0\rho}{Q_0}\frac{b_1}{2},$$
$$E=T_0\frac{\alpha_0\rho}{Q_0}\frac{2\,a\,v_0}{c^2-v_0^2}-T_0\frac{\alpha_0\rho}{Q_0}\frac{1}{l}+T_0\frac{\alpha_0\rho}{Q_0}\,c_1, F=T_0\frac{\alpha_0\rho}{Q_0}.$$

The resistance coefficient is a determined as

$$a=\frac{16\,\nu}{d^2}. \tag{11}$$

Temperature dependence of kinematic viscosity in the first approximation will be taken linearly

$$\nu=\nu_0+\frac{\nu_T-\nu_0}{T_0-T_c}\left(T_0-T\right). \tag{12}$$

Then from expression (11), taking into account formula (12), we obtain

$$a=\frac{16}{d^2}\left[\nu_0+\frac{\nu_T-\nu_0}{T_0-T_c}\left(T_0-\left(1-\frac{1}{3}a_1x^3-\frac{1}{2}b_1x^2-c_1x\right)\times\right.\right.$$
$$\left.\left.\times\left(\frac{1}{6}A\,x^6+\frac{1}{5}B\,x^5+\frac{1}{4}C\,x^4+\frac{1}{3}D\,x^3+\frac{1}{2}E\,x^2+F\,x+T_{\text{inlet}}\right)\right)\right]. \tag{13}$$

The equation of unsteady fluid motion will be obtained from expression (1) and will have the form:

$$\frac{\partial^2 P}{\partial t^2}=\left(c^2-v_0^2\right)\frac{\partial^2 P}{\partial x^2}-2v_0\frac{\partial^2 P}{\partial x\,\partial t}-2\,a\,v_0\frac{\partial P}{\partial x}-2\,a\frac{\partial P}{\partial t}+2\,c^2Q_0\frac{\partial a}{\partial x}, \tag{14}$$

where v_0 is the speed of undisturbed flow through the pipe.

The solution of equation (14) satisfying the boundary conditions:

$$P|_{x=0}=P_{\text{inlet}}\left(t\right), P|_{x=l}=P_c\left(t\right), \tag{15}$$

we will search in the form

$$P=P_{\text{inlet}}\left(t\right)-\frac{P_{\text{inlet}}\left(t\right)-P_c\left(t\right)}{l}x+\sum_{i=1}^{\infty}\varphi_i\left(t\right)\sin\frac{i\pi x}{l}, \tag{16}$$

where $\varphi_i\left(t\right)$ is unknown time-dependent function to be determined.

Substituting expression (16) into equation (14) and applying the Galerkin method, taking into account expression (13), we find the expression for $\varphi_i\left(t\right)$, and then $P\left(x,\,t\right)$.

Similarly, the pressure field is determined in other sections of the pipeline.

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Movement of a gas-liquid mixture in a vertical pipe

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Consider the movement of the gas-liquid mixture in the column of lifting pipes. Then, assuming the gas-liquid mixture is homogeneous with density ρ_{mix}, for the equation of its motion in the central channel and the continuity equation, we will have [1-3]

$$-\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial t} + 2aQ + \rho_{mix} g, -\frac{1}{c^2}\frac{\partial P}{\partial t} = \frac{\partial Q}{\partial x} \tag{1}$$

where, $c^2 = \frac{\partial}{\partial \rho}$, c- is the speed of sound in the liquid, $Q = \rho u$ is the mass flow rate of the liquid per unit area of the cross-section of the pipe, ρ_{mix} is the density of the mixture, u- is the mixture flow velocity averaged over the cross-section of the channel (pipe), a-is the drag coefficient, x-is the coordinate, t -is the time.

The density of the mixture ρ_{mix} can be found by the formula [3]

$$\rho_{mix} = \frac{(1+\eta)\rho_{atm}}{P_{atm}} \; . \tag{2}$$

Here η is the mass fraction of oil in the gas, is the density of the gas at atmospheric pressure, P_{atm} is atmospheric pressure, P is the pressure of the mixture in any cross-section of the pipe. Substituting expression (2) into the system (1) and after that, we differentiate the first equation of the system with respect to the coordinate x and the second with respect to time t, we subtract one from the other and obtain

$$\frac{\partial^2 P}{\partial t^2} = c^2\frac{\partial^2 P}{\partial x^2} - 2a\frac{\partial P}{\partial t} + g(1+\eta)\frac{\partial P}{\partial x}. \tag{3}$$

By placing the origin of the coordinate axis at the lower end of the pipe and pointing it upwards, for the initial and boundary conditions we will have

$$P|_{t=0} = f(x); \tag{4}$$

$$\left.\frac{\partial P}{\partial t}\right|_{t=0} = 0\,;\; P|_{x=0} = P_w(t); \tag{5}$$

$$P|_{x=l} = P_{wh}(t). \tag{6}$$

$f(x)$ a function that determines the pressure field of the gas-liquid mixture along the axis with its stationary flow, $P_w(t)$,$P_{wh}(t)$ respectively, the pressure at the bottom and wellhead. For the stationary regime of the flow of a gas-liquid mixture, within the framework of the accepted assumptions, from expression (1) we will have

$$-\frac{dP}{dx} = 2aQ_0 + (1+\eta)\frac{\rho_{atm}}{P_{atm}}\, g. \tag{7}$$

Integrating equations (7) with the boundary condition (6) taken into account, we obtain

$$P = P_c(0)\exp\left[-(1+\eta)\frac{\rho_{atm}}{P_{atm}}gx\right] - \frac{2aQ_0}{(1+\eta)\frac{\rho_{atm}}{P_{atm}}g} + 2aQ_0\frac{\exp\left[-(1+\eta)\frac{\rho_{atm}}{P_{atm}}gx\right]}{(1+\eta)\frac{\rho_{atm}}{P_{atm}}g}. \tag{8}$$

Expanding (8) into a Taylor series in the first approximation, taking into account only one term from expression (8), we obtain

$$f(x) = P(x) = P_w(0)\left[1-(1+\eta)\frac{\rho_{atm}}{P_{atm}}gx\right] - 2aQ_0 x. \tag{9}$$

The solution of equation (3), taking into account the boundary conditions (6) will be sought in the form. $\varphi_i(t)$ is an unknown function dependent time-t.

$$P = P_w(t) - \frac{P_w(t)-P_{wh}(t)}{l}x + \sum_{i=1}^{\infty}\varphi_i(t)\sin\frac{i\pi x}{l}. \tag{10}$$

Substituting expression (10) into equation (3), multiplying both parts of the resulting expression by $\sin\frac{i\pi x}{l}$ and integrating it from 0 to l, applying the Laplace transform, we get:

$$\overline{\varphi_i} = \frac{s}{(s+a)^2+\omega_i^2}\varphi_i(0) + \frac{1}{(s+a)^2+\omega_i^2}\dot{\varphi}_i(0) + \frac{1}{(s+a)^2+\omega_i^2}2a\varphi_i(0)+$$
$$+\frac{\frac{2}{i\pi}((-1)^i\overline{\ddot{P}}_{wh}-\overline{\ddot{P}}_w)}{(s+a)^2+\omega_i^2} + \frac{\frac{4a}{i\pi}((-1)^i\overline{\dot{P}}_{wh}-\overline{\dot{P}}_w)}{(s+a)^2+\omega_i^2} -$$
$$-\frac{2}{i\pi}(1-(-1)^i)g\frac{\frac{\bar{P}_w-\bar{P}_{wh}}{l}(1+\eta)}{(s+a)^2+\omega_i^2}. \tag{11}$$

$\omega_i^2 = \frac{\pi^2 c^2}{l^2} - a^2$, $\varphi_i(0)$ and $\dot{\varphi}_i(0)$ is determined from the initial conditions (4) and (5). From expressions (9) and (10) we have

$$P_w(0)\left[1-(1+\eta)\frac{\rho_{atm}}{P_{atm}}gx\right]-2aQ_0x = P_w(0)-\frac{P_w(0)-P_{wh}(0)}{l}x+\sum_{i=1}^{\infty}\varphi_i(0)\sin\frac{i\pi x}{l} \quad (12)$$

Multiplying both sides of equation (12) by $\sin\frac{i\pi x}{l}$ and integrating it from 0 to l, we get:

$$\varphi_i(0)=\frac{2l}{i\pi}(-1)^i(1+\eta)\frac{\rho_{atm}}{P_{atm}}gP_c(0)+\frac{2l}{i\pi}(-1)^i 2aQ_0-\frac{2l}{i\pi}(-1)^i\left[P_w(0)-P_{wt}(0)\right]. \quad (13)$$

Substituting expression (10) into the initial condition (11), multiplying the resulting expression by $\sin\frac{i\pi x}{l}$and integrating it from 0 to l, we obtain:

$$\dot{\varphi}_i(0)=\frac{2}{i\pi}\left[(-1)^i\dot{P}_{wh}(0)-\dot{P}_w(0)\right]. \quad (14)$$

To determine the mass flow rate of liquid in the central channel, substitute formulas (1) and (10) into the first equation of system (1) and then apply the Laplace transform to the resulting expression, taking into account formula (11), we obtain

$$\overline{Q}=\frac{Q_0}{s+2a}-\frac{\overline{P}_w(t)-\overline{P}_{wh}(t)}{l(s+2a)}+\sum_{i=1}^{\infty}\frac{\overline{\varphi}_i(t)}{s+2a}\frac{i\pi}{l}\cos(\frac{i\pi x}{l})-$$

$$-g\frac{\rho_{atm<}}{P_{atm}}(1+\eta)\left[\frac{\overline{P}_c(t)}{s+2a}-\frac{x}{l}\frac{\overline{P}_w(t)-\overline{P}_{wh}(t)}{s+2a}\sum_{i=1}^{\infty}\frac{\overline{\varphi}_i(t)}{s+2a}\frac{i\pi}{l}\sin(\frac{i\pi x}{l})\right]. \quad (15)$$

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Displacement of a string from finitely many points under the action of centralized forces in stationary mode

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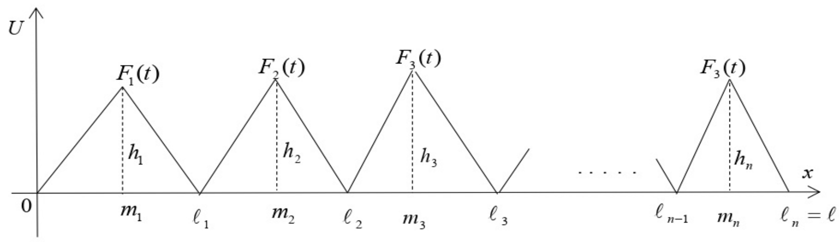
In the paper we consider a problem on determining the displacement of the string of length ℓ and whose initial condition is negligibly small from infinitely many points under the action of centralized forces [3, 6].

The replacing force of centralized forces acting on the points $x = m_1,\ x = m_2, ..., x = m_n$ of the string is reduced to the group of problems with no initial condition depending on the boundary mode. During the vibration process, the long-term action of boundary mode minimizes the action of initial conditions. In the absence of friction, the problem need not be solved. But consideration of friction action makes it convenient to choose the centralized of force in the form of time-dependent Fourier series in the form of linear combination of trigonometric functions that change by harmonic law in the stationary mode [1,2,4].



Problem statement. Writion boundary and adjoint conditions

$$\frac{\partial^2 U}{\partial t^2} = a^2 \frac{\partial^2 U}{\partial x^2}, \tag{1}$$

$$U_1(0,t) = 0\,,\ U_2(\ell,t) = 0, ..., U_n(\ell_n,t) = 0 \tag{2}$$

$$U_1(m_1,t) = U_2(m_1,t)\,,\ U_2(m_2,t) = U_3(m_2,t)\,, ..., U_{n-1}(m_n,t) = U_n(m_n,t) \tag{3}$$

as a result of action of the force

$$F(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos\omega_n t + B_n \sin\omega_n t)\,, \tag{4}$$

We determine the displacement.Here $F(t) = \sum_{n=1}^{n} F_n(t)$ is a replacing force.

Problem solution. In the absence of friction, none of the frequencies ω_n cndiderr with natural frequency of the string connected to the axis Ox from its ends [1, 2, 5].

$$U(x,t) = \frac{A_0}{2\ell}x + \sum_{n=1}^{\infty}(A_n \cos\omega_n t + B_n \sin\omega_n t)\frac{\sin\frac{\omega_n}{a}x}{\sin\frac{\omega_n}{a}\ell}, \tag{5}$$

It is convenient to relate the continuity of the sought-for displacement function (2)-(3) $U(x,t)$, where homogeneous boundary and adjointness are satisfied together, at the points $x = m_1,\ x = m_2, ..., x = m_n$ and also discontinuity of derivative among special solutions with centralized force $F_1(t),\ F_2(t), ..., F_n(t)$.

$$\left.\begin{array}{l} \frac{\partial U}{\partial x}\Big|_{m_1-0}^{m_1+0} = \frac{\partial U_2}{\partial x}(m_1,t) - \frac{\partial U_1}{\partial x}(m_1,t) = -\frac{F_1(t)}{k}, \\ \frac{\partial U}{\partial x}\Big|_{m_2-0}^{m_2+0} = \frac{\partial U_3}{\partial x}(m_2,t) - \frac{\partial U_2}{\partial x}(m_2,t) = -\frac{F_2(t)}{k}, \\ ., \\ \frac{\partial U}{\partial x}\Big|_{m_n-0}^{m_n+0} = \frac{\partial U_n}{\partial x}(m_n,t) - \frac{\partial U_{n-1}}{\partial x}(m_n,t) = -\frac{F_n(t)}{k} \end{array}\right\} \tag{6}$$

Special solutions of a string formed as a result of the action of a force that changes in time and harmonically at a nodal point in the interval $t < +\infty$ for a stationary mode are determined by means of n number various solutions i.e.

$$\left.\begin{array}{l} U(x,t) = U_1(x,t),\ 0 \le x \le m_1, \\ U(x,t) = U_2(x,t),\ m_1 \le x \le \ell_1, \\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \\ U(x,t) = U_{n-1}(x,t),\ \ell_{n-1} \le x \le m_n, \\ U(x,t) = U_n(x,t),\ m_n \le x \le \ell_n = \ell \end{array}\right\} \tag{7}$$

Here, using the adjointness conditions, we obtain n number special solutions

$$U(x,t) = \left\{\begin{array}{l} U_1(x,t) = \frac{Aa}{k\omega_1}\cdot\frac{\sin\frac{\omega_1}{a}(\ell_1 - m_1)}{\sin\frac{\omega_1}{a}\ell_1}\sin\frac{\omega_1}{a}x\cdot\cos\omega_1 t,\ 0 \le x \le m_1, \\ U_2(x,t) = \frac{Aa}{k\omega_2}\cdot\frac{\sin\frac{\omega_2}{a}m_1}{\sin\frac{\omega_2}{a}\ell_1}\sin\frac{\omega_2}{a}(\ell_1 - x)\cos\omega_2 t,\ m \le x \le \ell_1, \\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \\ U_n(x,t) = \frac{Aa}{k\omega_n}\cdot\frac{\sin\frac{\omega_n}{a}m_n}{\sin\frac{\omega_n}{a}\ell_n}\sin\frac{\omega_n}{a}(\ell_n - x)\cdot\cos\omega_n t,\ m_n \le x \le \ell_n = \ell, \end{array}\right. \tag{8}$$

When the fraction factor is $\alpha \neq 0$ in the stationary mode the displacement created as a result of periodically changing forces (4) is given in the form of the linear combination of the functions

$$U_1(x,t),\ U_2(x,t), ..., U_{n-1}(x,t),\ U_n(x,t)$$

i.e.

$$U(x,t)=\begin{cases} U_1(x,t)=\frac{1}{k}\left[\frac{A_0x}{2}\left(1-\frac{m_1}{\ell_1}\right)+\sum_{n=1}^{\infty}\frac{a\sin\frac{\omega_n}{a}(\ell_1-m_1)}{\omega_n\sin\frac{\omega_n}{a}\ell_1}\right]\sin\frac{\omega_n x}{a}\times\\ \times(A_n\cos\omega_n t+B_n\sin\omega_n t),\ \ 0\le x\le m_1.\\ U_2(x,t)=\frac{1}{k}\left[\frac{A_0m_1}{2}\left(1-\frac{x}{\ell_1}\right)+\sum_{n=1}^{\infty}\frac{a\sin\frac{\omega_n}{a}m_1}{\omega\sin\frac{\omega_n}{a}\ell_1}\right]\sin\frac{\omega_n(\ell_1-x)}{a}\times\\ \times(A_n\cos\omega_n t+B_n\sin\omega_n t),\ \ m_1\le x\le \ell_1.\\ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\\ U_{n-1}(x,t)=\frac{1}{k}\left[\frac{A_0\ell_{n-1}}{2}\left(1-\frac{m_1}{\ell_{n-1}}\right)+\sum_{n=1}^{\infty}\frac{a\sin\frac{\omega_n}{a}(\ell_{n-1}-m_n)}{\omega_n\sin\frac{\omega_n}{a}\ell_{n-1}}\right]\times\\ \times\sin\frac{\omega_n x}{a}(A_n\cos\omega_n t+B_n\sin\omega_n t),\ \ \ell_{n-1}\le x\le m_n,\\ U_n(x,t)=\frac{1}{k}\left[\frac{A_0m_n}{2}\left(1-\frac{x}{\ell_n}\right)+\sum_{n=1}^{\infty}\frac{a\sin\frac{\omega_n}{a}m_n}{\omega_n\sin\frac{\omega_n}{a}\ell_n}\right]\sin\frac{\omega_n(\ell_n-x)}{a}\times\\ \times(A_n\cos\omega_n t+B_n\sin\omega_n t),\ \ m_n\le x\le \ell_n=\ell. \end{cases}\qquad(9)$$

When applying the centralized forces at the points $m_1,\ m_2,...,m_n$ if appropriate frequencies are $\omega_1,\ \omega_2,...,\ \omega_n$ free wibrations of the string in the nodes of stagnant waves are in the following form:

$$\sin\frac{\omega_1}{a}m_1=0,\quad \sin\frac{\omega_1}{a}(\ell_1-m_1)=0,\ ...,\sin\frac{\omega_n}{a}m_n=0,\quad \sin\frac{\omega_n}{a}(\ell_n-m_n)=0$$

Conclusion. Vibrations of a string stretched from finitely many nodal points under the action of centralized forces in the stationary mode, are based on the superposition principle of waves that vibrate as a result of the action of replacing force that harmonically changes in time.

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On characterization of precompact sets on weighted Lebesgue spaces

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In this abstract we give a characterization an analog of the Kolmogorov-Riesz compactness criteria on weighted Lebesgue spaces by weighted modulus of continuity.

Let $x \in (0, \infty)$ and let $\rho(x) = 1 + x^2$.

$$\Omega_p(f;\delta) = \sup_{0<h\leq\delta} \left(\int_0^\infty \left(\frac{|f(x+h) - f(x)|}{(1+x^2)(1+h^2)} \right)^p dx \right)^{\frac{1}{p}}$$

We refer reader to the works [1] and [2].

Theorem 2. Let $1 \leq p < \infty$ and let F be a family in $L_{p,\rho}(0, \infty)$. Then, F is totally bounded in $L_{p,\rho}(0, \infty)$, if and only if the following conditions are satisfied:

1. F is bounded in $L_{p,\rho}(0, \infty)$, i.e., there exists $M > 0$ such that $\sup\limits_{f \in F} \|f\|_{L_{p,\rho}}(0, \infty) \leq M$,

2. for every $\varepsilon > 0$ there is some $\delta > 0$ so that, for every $f \in F$ and $0 < h < \delta$
$$\Omega_p(f, h) < \varepsilon,$$

3. for every $\varepsilon > 0$ there is some $\eta > 0$ so that, for every $f \in F$
$$\|f\|_{Lp,\rho(x>\eta)} < \varepsilon.$$

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Optimal control problem for a $3D$ Bianchi type integro-differential equation with non-classical boundary conditions

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Optimal control problems described by hyperbolic equations under Goursat conditions originate from the works of A.I. Egorov [3]. In this paper, we study one problem of optimal control described with 3D Bianchi type integro-differential equations under non-classical boundary conditions. It should be especially noted that the 3D Bianchi type integro-differential equations are one of the main differential equations of mathematical physics [4]. For the optimal control problem under consideration, necessary optimality conditions are found in the form of the Pontryagin maximum principle.

Let the controlled object be described by the 3D (three-dimensional) Bianchi type integro-differential equation:

$$(V_{1,1,1}u)(x,y,z) \equiv u_{xyz}(x,y,z) + A_{0,0,0}u(x,y,z) + A_{1,0,0}u_x(x,y,z)+$$

$$+A_{0,1,0}u_y(x,y,z) + A_{0,0,1}u_z(x,y,z) + A_{1,1,0}u_{xy}(x,y,z) + A_{0,1,1}u_{yz}(x,y,z)+$$

$$+A_{1,0,1}u_{xz}(x,y,z) + \int\limits_{\sqrt{x_0x_1}}^{x}\int\limits_{\sqrt{y_0y_1}}^{y}\int\limits_{\sqrt{z_0z_1}}^{z}[K_{0,0,0}(\tau,\xi,\eta;x,y,z)\ u(\tau,\xi,\eta) + K_{1,0,0}(\tau,\xi,\eta;x,y,z)\times$$

$$\times u_x(\tau,\xi,\eta) + K_{0,1,0}(\tau,\xi,\eta;x,y,z)u_y(\tau,\xi,\eta) + K_{0,0,1}(\tau,\xi,\eta;x,y,z)\times$$

$$\times u_z(\tau,\xi,\eta) + K_{1,1,0}(\tau,\xi,\eta;x,y,z)u_{xy}(\tau,\xi,\eta + K_{0,1,1}(\tau,\xi,\eta;x,y,z)\times$$

$$\times u_{yz}(\tau,\xi,\eta) + K_{1,0,1}(\tau,\xi,\eta;x,y,z)u_{xz}(\tau,\xi,\eta)]\,d\tau d\xi d\eta = \varphi(x,y,z,v(x,y,z)),$$

$$(x,y,z) \in G = (x_0,x_1)\times(y_0,y_1)\times(z_0,z_1) \tag{1}$$

under the following non-classical 3D boundary conditions in the geometric middle of the domain:

$$\begin{cases} V_{0,0,0}u \equiv u(\sqrt{x_0x_1}, \sqrt{y_0y_1}, \sqrt{z_0z_1}) = \varphi_{0,0,0} \\ (V_{1,0,0}u)(x) \equiv u_x(x, \sqrt{y_0y_1}, \sqrt{z_0z_1}) = \varphi_{1,0,0}(x), \\ (V_{0,1,0}u)(y) \equiv u_y(\sqrt{x_0x_1}, y, \sqrt{z_0z_1}) = \varphi_{0,1,0}(y), \\ (V_{0,0,1}u)(z) \equiv u_z(\sqrt{x_0x_1}, \sqrt{y_0y_1}, z) = \varphi_{0,0,1}(z), \\ (V_{1,1,0}u)(x,y) \equiv u_{xy}(x, y, \sqrt{z_0z_1}) = \varphi_{1,1,0}(x,y), \\ (V_{0,1,1}u)(y,z) \equiv u_{yz}(\sqrt{x_0x_1}, y, z) = \varphi_{0,1,1}(y,z), \\ (V_{1,0,1}u)(x,z) \equiv u_{xz}(x, \sqrt{y_0y_1}, z) = \varphi_{1,0,1}(x,z), \end{cases} \tag{2}$$

where $D_\tau = \partial/\partial\tau$ is a generalized differentiation operator in the Sobolev sense; $u(x,y,z)$ is the desired function; $K_{i,j,k}(\tau,\xi,\eta;x,y,z) \in L_\infty(G \times G)$; $A_{i,j,k}(x,y,z)$are the given measurable functions on G. $\varphi(x,y,z,v)$ are the given functions on $G \times R^r$ satisfying the Caratheodory conditions; $v(x,y,z) = (v_1(x,y,z), \cdots, v_r(x,y,z))$ is r-dimensional controlling vector-function; $\varphi_{i,j,k}$ are the given elements.

Let the function $v(x,y,z)$ measurable and bounded on G and almost at all points of $(x,y,z) \in G$ accept its values from some given set $\Omega \subset R^r$. Then this vector function is said to be an admissible control. A set of all admissible controls is denoted by Ω_∂.

In the paper we consider the following linear problem of optimal control: find admissible control $v(x,y,z)$ from Ω_∂, for which the solution

$$u \in W_p^{(1,1,1)}(G) \equiv \{\, u \in L_p(G) \,/ D_x^i\, D_y^j\, D_z^k u \in L_p(G);\ \ i,j,\ k = 0,\, 1\}$$

$(1 \le p \le \infty)$ of 3D non-classical boundary-problem (1)-(2) delivers the least value to the 3D linear multi-point functional:

$$S(v) = \sum_{k=1}^{N} [a_k^{(1,0,0)} u(x_k^{(1)}, y_1, z_1) + a_k^{(0,1,0)} u(x_1, y_k^{(1)}, z_1) + a_k^{(0,0,1)} u(x_1, y_1, z_k^{(1)})] \to \min, \tag{3}$$

where $(x_k^{(1)}, y_k^1, z_k^1) \in \bar{G}$ are the given points; $a_k^{(1,0,0)}, a_k^{(0,1,0)}, a_k^{(0,0,1)} \in R$ are the given numbers.

The necessary and sufficient conditions for an optimal process for ordinary differential equations and partial differential equations under local conditions are sufficiently thoroughly studied by many mathematicians. The results obtained in this area were studied in detail in monographs such as [1] other. Also highlighted the work of [2] et al., which studied various classes of problems of optimum control.

The 3D optimal control problem (1)-(3) was studied by means of a new variant the increment method. The method essentially uses the notion of an integral form conjugation equation and allows also to cover the case when the coefficients of the equation (1) are, generally speaking, non-smooth functions. In other words, these variants are more natural than classic increment variants developed for example by [3].

Note that the optimal control problem (1)-(3) is investigated using the following 3D integral representation of modification variant in space $W_p^{(1,1,1)}(G)$:

$$u(x,y,z) = u(\sqrt{x_0x_1},\sqrt{y_0y_1},\sqrt{z_0z_1}) + \int_{\sqrt{x_0x_1}}^{x} u_\alpha(\alpha,\sqrt{y_0y_1},\sqrt{z_0z_1})d\alpha +$$
$$+\int_{\sqrt{y_0y_1}}^{y} u_\beta(\sqrt{x_0x_1},\beta,\sqrt{z_0z_1})d\beta +$$
$$+\int_{\sqrt{z_0z_1}}^{z} u_\gamma(\sqrt{x_0x_1},\sqrt{y_0y_1},\gamma)d\gamma + \int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y} u_{\alpha\beta}(\alpha,\beta,\sqrt{z_0z_1})d\alpha d\beta +$$
$$+\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} u_{\beta\gamma}(\sqrt{x_0x_1},\beta,\gamma)d\beta d\gamma + \int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{z_0z_1}}^{z} u_{\alpha\gamma}(\alpha,\sqrt{y_0y_1},\gamma)d\alpha d\gamma +$$
$$+\int_{\sqrt{x_0x_1}}^{x}\int_{\sqrt{y_0y_1}}^{y}\int_{\sqrt{z_0z_1}}^{z} u_{\alpha\beta\gamma}(\alpha,\beta,\gamma)d\alpha d\beta d\gamma.$$

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On the some classical inequalities for algebraic polynomials in complex plane

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Let $G \subset \mathbb{C}$ be a bounded Jordan region with boundary $L := \partial G$. Let $\wp_n$ denotes the class of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$; $h\left(z\right)$ be a generalized Jacobi weight function.

Let $0 < p \leq \infty$. For the arbitrary Jordan region G, we introduce:

$$\begin{aligned} \left\| P_n \right\|_{A_p(h,G)} &= \left(\iint\limits_G h(z) \left| P_n(z) \right|^p d\sigma_z \right)^{1/p}, \ 0 < p < \infty, \\ \left\| P_n \right\|_{A_\infty(1,G)} &= \max_{z \in \overline{G}} \left| P_n(z) \right|, \ p = \infty; \end{aligned}$$

and when L is rectifiable:

$$\begin{aligned} \left\| P_n \right\|_{\mathcal{L}_p(h,L)} &:= \left(\int\limits_L h(z) \left| P_n(z) \right|^p |dz| \right)^{1/p} < \infty, 0 < p < \infty, \\ \left\| P_n \right\|_{\mathcal{L}_\infty(1,L)} &:= \left\| P_n \right\|_{A_\infty(1,G)}, \ p = \infty. \end{aligned}$$

In this work, we study the following type of estimates

$$\|P_n^{(m)}\|_X \leq \lambda_n \|P_n\|_Y \tag{1}$$

for some spaces X and Y, where $\lambda_n > 0$, $\lambda_n \to \infty$, $n \to \infty$, is a constant, depending on the geometrical properties of the curve L, the weight function h and spaces X, Y. In the literature, these inequalities are often called Bernstein-type for $X = Y = \mathcal{A}_\infty(1,G)(\mathcal{L}_\infty(1,L))$; Markov-type for $X = Y = \mathcal{A}_p(h,G)(\mathcal{L}_p(h,L))$, $p > 0$, and Nikolskii-type for $m = 0$, $X = A_q(h,G)(\mathcal{L}_q(h,L)), Y = A_p(h,G)(\mathcal{L}_p(h,L))$, $0 < p < q < 1$, inequalities in Bergman and Lebesgue spaces for a polynomials $P_n \in \wp_n$ and $m = 0, 1, 2, , , .$

Inequalities of type [1] for some $m \geq 0$, X, Y, $h(z)$, G and $0 < p \leq \infty$, have been studied by many mathematicians since the beginning of the

20th century (Bernstein SN., Szegö G & Zygmund A.). Over the past few years, such inequalities for various spaces have been studied by, for example, Nikol'skii SM., Milovanovic GV & Mitrinovic DS. & Rassias ThM., Dzjadyk VK, Andrashko, Kabayla V., Mamedkhanov DI., Pritsker I., Batchaev IM., Andrievskii VV., Ditzian Z & Tikhonov S., Ditzian Z & Prymak A., Nevai P & TotikV., Andrievskii VV, Abdullayev FG. (see also the references cited therein) and others.

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An approach to the analysis and classification of medical errors

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Errors of medical personnel have a particularly negative impact on the ability to exercise vital rights, freedoms, and legal interests of a person. The contradictions about the quality of the provided medical services show that people who need medical help often become hostages of the prevailing stereotype about the objective inevitability of medical errors. Specialists who harm the patient, in order to avoid moral and legal responsibility, show their mistakes as one of the completely natural directions in the development of the current professional situation.

Only in recent years, steps have been taken to eliminate this problem. Thus, in the first step, three main strategies were proposed to improve patient safety:

1) error reporting by the security management system; 2) error learning and investigation; 3) establishing fair information sharing in hospitals and physicians' outpatient practices.

If these are done, critical incidents are identified, reported, and analyzed so that similar incidents can be prevented and action taken. Every incident must be evaluated. Finally, if avoidable adverse events ever occur, it must take appropriate action to prevent damage.

Based on the context accepted today, we can divide medical errors into the groups shown in table 1.

Table 1 . Overview of medical malpractice

No	Event	Explanation
1	Adverse event	A further treatment-induced adverse event may or may not be preventable
2	A preventable adverse event	That negative event that can be prevented
3	Critical event	An event resulting in an adverse event or an event with an increased likelihood of occurrence
4	error	Action or inaction that leads to deviation from the goal, following a wrong plan, lack of planning.
5	Avoid partial review	A mistake that does not cause much damage

A doctor's error, regardless of whether it is big or small, should be investigated and evaluated from both a quantitative and a qualitative point of view. τ_0 Evaluate his work performance

$$A = A\left\{N;\ N_i;\ N_t\ \ (i \geq t)\,;D_i^m;\ Y_i^m;\ Q_i^m\right\} \tag{1}$$

is equivalent to evaluating the functional.

Here A is the doctor of the emergency brigade; N is the number of working days of the doctor in a month; N_i is one working day of the doctor; $N_t\ \ (i \geq t)$ is the number of the doctor's direct participation in the call (except for cases of false calls, refusal of the doctor, patient's absence at the address of the call); D_i^m is initial diagnosis made by the doctor; Y_i^m is the provided first aid; Q_i^m is decision-making by the doctor.

But it should not be forgotten that problems are inevitable in the work of a doctor, especially in the work of emergency doctors. For example, the information about the calls of the Emergency and Urgent Medical Aid Station in Baku city in 2020-2021 is given in table 2. Based on the data, it can be seen what types of calls doctors face every day.

Table 2 . Events encountered by the medical team

No	**Call events**	**2020 year**		**2021 year**	
		During the year	**1 day**	**During the year**	**1 day**
1	Inconclusive calls	63555	175	62424	171
2	False call	1015	3	1206	4
3	Not found in place	6206	17	6310	17
4	Address not found	749	2	1076	3
5	Death to the doctor	16044	44	16852	46
6	Unreasonable call	39541	108	36980	101

The effectiveness of the organization of medical assistance to the population mainly depends on the extent to which the regional characteristics affecting it are adequately taken into account. Therefore, it is necessary to analyze the activity of ambulatory and emergency medical services at the regional level, and then develop measures aimed at improving their work. We believe that the role of modern information technologies in the application of evidence-based measures and methods is undeniable.

Based on the doctors' questionnaire, a survey table was developed for the analysis of the objective reasons for discrepancies in diagnoses (percentage of positive answers).

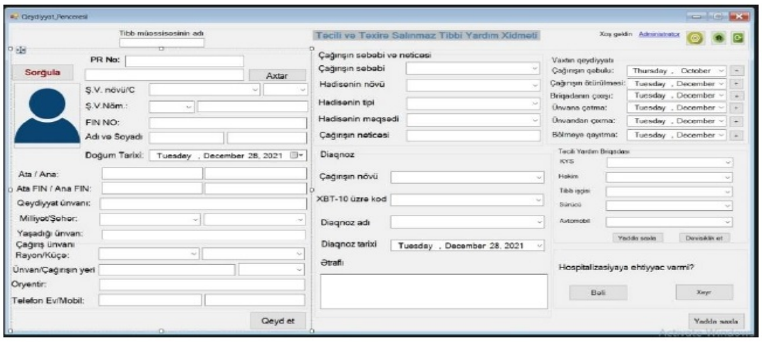

Fig. 1. Fragment from the doctor's tablet

Doctors of the ambulatory, polyclinic, emergency, and urgent care organizations are asked to answer the following types of questions: lack of information to assess the patient's condition, in some cases incompleteness of diagnostic equipment and medical equipment, hidden form of the disease, rarity or severe course of the disease.

Underestimation or overestimation of the data obtained from laboratory, instrumental, and other additional research methods can also harm the correct diagnosis.

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Weakly periodic p-adic generalized Gibbs measures for the Ising model on the Cayley tree of order two

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In the present paper, we study the $p-$adic Ising model on the Cayley tree of order two. The existence of H_A-weakly periodic (non-periodic) $p-$adic generalized Gibbs measures for this model is shown.

Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime p, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, and m and n are relatively prime with p, r is called the order of x and written $r = ord_p x$. The $p-$adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

This norm is non-Archimedean, i.e., satisfies the strong triangle inequality $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}$.

The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field $\mathbb{Q}_p$ (see [1]).

The completion of the field of rational numbers $\mathbb{Q}$ is either the field of real numbers $\mathbb{R}$ or one of the fields of p-adic numbers $\mathbb{Q}_p$ (Ostrowski's theorem).

Any p-adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...)$$

where $\gamma(x) \in \mathbb{Z}$ and the integers x_j satisfy: $x_0 \neq 0,\ x_j \in \{0, 1, ..., p-1\}$, $j \in \mathbb{N}$ (see [1]). In this case $|x|_p = p^{-\gamma(x)}$.

Let us consider the following set on the Cayley tree $\Gamma^k(V, L)$ (see [2]). Let $x_o \in V$ be fixed,

$$W_n = \{x \in V : |x| = n\}, \quad V_n = \{x \in V : |x| \leq n\},$$

$$L_n = \{l = \langle x, y \rangle \in L : x, y \in V_n\},$$

$$S(x) = \{y \in V : x \to y\}, \quad S_1(x) = \{y \in V : d(x, y) = 1\}$$

and for $x \in W_n$, denote $S(x) = \{y \in W_{n+1} : d(x,y) = 1\}$. The set is called direct successors of x. The set $S(x)$ is called the set of direct successors of the vertex x.

We consider p-adic Ising model on the Cayley tree Γ^k. Let $\mathbb{Q}_p$ be a field of p-adic numbers and $\Phi = \{-1, 1\}$. A configuration σ on V is defined by the function $x \in V \to \sigma(x) \in \Phi$. Similarly, one can define the configuration σ_n and $\sigma^{(n)}$ on V_n and W_n, respectively. The set of all configurations on V (resp. V_n, W_n) is denoted by $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$).

For given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\varphi^{(n)} \in \Omega_{W_n}$ we define a configuration in Ω_{V_n} as follows

$$(\sigma_{n-1} \vee \varphi^{(n)})(x) = \begin{cases} \sigma_{n-1}(x), & \text{if } x \in V_{n-1}, \\ \varphi^{(n)}(x), & \text{if } x \in W_n. \end{cases}$$

A formal p-adic Hamiltonian $H : \Omega \to \mathbb{Q}_p$ of the p-adic Ising model is defined by

$$H(\sigma) = J \sum_{<x,y> \in L} \sigma(x)\sigma(y),$$

where $0 < |J|_p < p^{-1/(p-1)}$ for any $\langle x, y \rangle \in L$.

We define a function $h : x \to h_x$, $\forall x \in V \setminus \{x_0\}$, $h_x \in \mathbb{Q}_p$ and consider p-adic probability distribution $\mu_h^{(n)}$ on Ω_{V_n} defined by

$$\mu_h^{(n)}(\sigma_n) = \frac{1}{Z_n^{(h)}} \exp_p\{H_n(\sigma_n)\} \prod_{x \in W_n} h_{\sigma(x),x} \quad n = 1, 2, \ldots,$$

where $Z_n^{(h)}$ is the normalizing constant

$$Z_n^{(h)} = \sum_{\varphi \in \Omega_{V_n}} \exp_p\{H_n(\varphi)\} \prod_{x \in W_n} h_{\sigma(x),x}.$$

A p-adic probability distribution $\mu_h^{(n)}$ is said to be consistent if for all $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, we have

$$\sum_{\varphi \in \Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \varphi) = \mu_h^{(n-1)}(\sigma_{n-1}).$$

In this case, by the p-adic analogue of the Kolmogorov theorem there exists a unique measure μ_h on the set Ω such that $\mu_h\left(\{\sigma|_{V_n} \equiv \sigma_n\}\right) = \mu_h^{(n)}\left(\sigma_n\right)$ for all n and $\sigma_n \in \Omega_{V_n}$.(see [3])

Theorem 1. *If $p \equiv 1(\mathrm{mod}\,4)$ then there exists at least two weakly periodic (non-periodic) p-adic generalized Gibbs measures for the Ising model on the Cayley tree of order two.*

Remark. *In [4] it was proved that for the Ising model on a Cayley tree of order $k = 2$ with respect to the normal divisor of index 2, there does not exist a weakly periodic (non-translation-invariant) Gibbs measure in the real case. In the p-adic case Theorem 1 was shown that for the Ising model, there are at least two new weakly periodic p-adic generalized Gibbs measures.*

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Criterion for completeness and minimality of the weighted exponential system with excess in the grand Lebesgue spaces

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The paper considers the exponential system $\{\rho(t)e^{int}\}_{n\in Z}$ with degenerate coefficient $\rho(\cdot)$ in grand Lebesgue space $G_{p)}(-\pi,\pi), 1<p<+\infty$. The completeness and minimality of such systems in classical Lebesgue spaces are sufficiently well studied in works of various mathematics [1-5]. The criteria for completeness and minimality of this system are established in $G_{p)}(-\pi,\pi), 1<p<+\infty$.

The grand Lebesgue space $L_{p)}(-\pi,\pi), 1<p<+\infty$ is nonseparable and therefore, we define the subspace $G_{p)}(-\pi,\pi)\subset L_{p)}(-\pi,\pi)$ in which the infinitely differentiable functions are dense. Consider the system

$$\{\rho(t)e^{int}\}_{n\in\mathbb{Z}}, \tag{1}$$

with the weight coefficient of the form

$$\rho(t)=\prod_{k=0}^{r}|t-t_k|^{\alpha_k},\ \alpha_k\in\mathbb{R},\ t_k\in[-\pi,\pi),\ k=\overline{0,r},$$

where $\{t_k\}_0^r\subset[-\pi,\pi)$ are distinct numbers.

Theorem 1. *The system (1) is complete in the space* $G_{p)}(-\pi,\pi), 1<p<+\infty$, *if and only if*

$$\alpha_k>-\frac{1}{p},\ k=\overline{0,r}.$$

Theorem 2. *The system (1) is minimal in the space* $G_{p)}(-\pi,\pi), 1<p<+\infty$, *if and only if*

$$-\frac{1}{p}<\alpha_k<\frac{1}{q},\ k=\overline{0,r},\ \frac{1}{p}+\frac{1}{q}=1.$$

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GeoGebra program skills in high schools (X-XI Grades) in Azerbaijan

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ICT should help the student to acquire better knowledge necessary for the success of the main state exam in mathematics, aimed at:
- formation of students' basic competencies in the process of learning and extracurricular activities;
- increase the learning motivation of students;
- acquisition of computer literacy by students, increasing the level of computer literacy of teachers;
- organization of independent and research activities of students;
- creation of a personal bank of ready-made educational and methodological resources in the educational process;
- development of spatial thinking and cognitive abilities of students.

Thus, the use of information technology in the classroom and in extracurricular activities expands the creative abilities of both teachers and students, increases interest in the subject, stimulates students to study fairly serious topics in computer science, which ultimately leads to the intensification of the educational process.

In this short part of our article, at the end of the article, we talk about some programs that are not mentioned (for example, GeoGebra) or are very little mentioned in our country to achieve this, which would be very useful in the Covid19 era.

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Fundamental solution for a nonlocal second order hyperbolic side problem

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In this paper second order equations system

$$(V_{1,1}z)(t,x) \equiv z_{tx}(t,x) + z(t,x)A_0(t,x) + z_x(t,x)A_1(t,x) +$$
$$+z_t(t,x)A_2(t,x) + \int_T z_\tau(\tau, h(t,x))K(\tau;t,x)\,d\tau = g_3(t,x),$$
$$(t,x) \in D = T \times X \quad T = [t_0,t_1]\,, X = [x_0,x_1]\,, \tag{1}$$

is considered under nonlocal boundary conditions [1, 2]

$$(V_{1,0}z)(t) \equiv \sum_{j=1}^{m} [z_t(t,\xi_j)\alpha_j(t) + z(t,\xi_j)\beta_j(t)] = g_2(t), \;\; t \in T, \tag{2}$$

$$(V_{0,1}z)(x) \equiv z_x(t_0,x) = g_1(x), \quad x \in X, \tag{3}$$

$$V_{0,0}z \equiv z(t_0,x_0) = g_0\,. \tag{4}$$

Here: $A_0(t,x)\,, A_1(t,x)\,, A_2(t,x)$ are given $n \times n$ matrces, where $A_0 \in L_{p,n\times n}(D)$, i.e., with elements from $L_p(D)\,,\ 1 \le p \le \infty$; there exist such functions $a_{1,0} \in L_p(X)\,,\ a_{0,1} \in L_p(T)$ that $\left\|A_{1,0}(t,x)\right\| \le a_{1,0}(x)\,,\ \left\|A_{0,1}(t,x)\right\| \le a_{0,1}(t)$ a.e. on D; $K(\tau;t,x)$ is given $n \times n$ matrix, such that $K(\cdot\,;t,x) \in L_{q,n\times n}(T)$ for almost every $(t,x) \in D$, $q = p/(p-1)$,moreover, the norm $\|K(\cdot\,;t,x)\|_{L_{q,n\times n}(T)}$ as function of $(t,x) \in D$ belongs to space $L_p(D)$; $h(t,x)$ is given measurable functions on D, for which $h(t,x) \in T$ for almost every $(t,x) \in D$; $g_3(t,x)\,, g_2(t)\,, g_1(x)\,, g_0$ are given line n- vectors, such that $g_3 \in L_{p,n}(D)\,,\ g_2 \in L_{p,n}(T)\,,\ g_1 \in L_{p,n}(X)$, i.e., with elements from $L_p(D)\,,\ L_p(T)$,$L_p(X)$, respectively; $\alpha_j(t)\,, \beta_j(t)$ - are given $n \times n$- matrces and $\alpha_j \in L_{\infty,n\times n}(T)\,,\ \beta_j \in L_{p,n\times n}(T)$; $\xi_j \in X\,,\ j = 1,...,m$ are given numbers.

Under abovementioned conditions on the data of the problem (1)-(4) we can assume its solution from Sobolev space

$W_{p,n}(D) = \{z \in L_{p,n}(D)\ ;\ z_t,\ z_x\ , z_{tx} \in L_{p,n}(D)\}$ [5] . In other words, the operator $V = (V_{0,0}, V_{0,1}, V_{1,0}, V_{1,1})$ has been defined on $W_{p,n}(D)$ and acts onto space $\Delta_{p,n}(D) = R^n \times L_{p,n}(X) \times L_{p,n}(T) \times L_{p,n}(D)$.

In this work there has been used an isomorphism which is implemented by the operator $Nz \equiv (z(t_0, x_0), z_x(t_0, x), z_t(t, x_0), z_{tx}(t, x))$ from $W_{p,n}(D)$ onto $\Delta_{p,n}(D)$ [2-4,7]. Basing on this isomorphism for the problem (1)-(4) there has been involved the concept adjoint problem in form of integro-algebraic system. Further using the special solutions of the adjoint problem here has been defined the concept of fundamental solution [3,4,6,7].

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On the basis property in $L_p\,(0,1)$ eigenfunctions of a second-order differential operator with a discontinuity point

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Consider the following spectral problem with a discontinuity point

$$y''(x)+\lambda y(x)=0,\quad x\in(0,\frac{1}{3})\cup(\frac{1}{3},\,1), \tag{1}$$

$$\begin{cases} y'(0)=y'(1)=0,\ y(\frac{1}{3}-0)=y(\frac{1}{3}+0), \\ y'(\frac{1}{3}-0)-y'(\frac{1}{3}+0)=\lambda m y(\frac{1}{3}), \end{cases} \tag{2}$$

where λ is a spectral parameter, and m is a non-zero complex number. Such spectral problems arise in the study of various problems of mathematical physics by the Fourier method [1].

The study of the basic properties of systems from eigenfunctions of spectral problems with a discontinuity point sometimes requires the use of other methods different from those previously known. In [2,3], a new method for studying the basic properties of discontinuous differential operators was proposed. In this paper, we study the basic properties of the eigenfunctions of problems (1), (2) by the method of [2,3] in weighted Lebesgue spaces.

The spectral problem (1),(2) has two series of eigenvalues:

$$\lambda_{1,n}=\left(\rho_{1,n}\right)^2$$

and

$$\lambda_{2,n}=\left(\rho_{2,n}\right)^2,\ n\in Z^+,$$

where $Z^+=N\cup\{0\}$, $\rho_{1,n}=3\pi n+\frac{3\pi}{2}+O\left(\frac{1}{n}\right),\rho_{2,n}=\frac{3\pi n}{2}+\frac{3\pi}{4}+O\left(\frac{1}{n}\right),$ and the corresponding eigenfunctions are given by the following expressions:

$$y_{i,n}(x)=\begin{cases}\cos\frac{2\rho_{i,n}}{3}\cos\rho_{i,n}x, & x\in\left[0,\frac{1}{3}\right],\\ \cos\frac{\rho_{i,n}}{3}\cos\rho_{i,n}(1-x), & x\in\left[\frac{1}{3},1\right],\end{cases}\qquad i=1,2;\ \ n\in Z^{+}. \quad (3)$$

Recall that the class of Mackenhoupt weights $A_p(I)$ is the class of periodic functions satisfying the condition

$$\sup_{J\subset I}\left(\frac{1}{|J|}\int_J \nu(t)\,dt\right)\left(\frac{1}{|J|}\int_I |\nu(t)|^{-\frac{1}{p-1}}\right)^{p-1}<+\infty,$$

where $I=(0,\,1)$ and the supremum is taken over all intervals $J\subset I$ and $|J|$ is a length of the interval J. Weighted Lebesgue space generated by the norm

$$\|f\|_{L_{p,\nu}(0,1)}=\left(\int_0^1 |f(x)|^p\,\nu(x)\,dx\right)^{\frac{1}{p}}.$$

denote by $L_{p,\nu}(0,1)$. We define the operator L in the space $L_{p,\nu}(0,1)\oplus C$ as follows. As the domain of definition $D(L)$ of the operator L, we take the manifold

$$D(L)=\left\{\hat{y}=\left(y(x),my\left(\tfrac{1}{3}\right)\right):y(x)\in W_p^2\left(0,\tfrac{1}{3}\right)\oplus W_p^2\left(\tfrac{1}{3},1\right),\right.$$
$$\left. y'(0)=y'(1)=0,\quad y\left(\tfrac{1}{3}-0\right)=y\left(\tfrac{1}{3}+0\right)\right\},$$

and for $\hat{y}\in D(L)$: $L\hat{y}=\left(-y'';\ y'\left(\frac{1}{3}-0\right)-y'\left(\frac{1}{3}+0\right)\right)$. The eigenvalues of the operator L are the numbers $\lambda_{i,n}$, and the corresponding eigenvectors have the form: $\hat{y}_{i,n}=\left(y_{i,n}(x),\ my\left(\frac{1}{3}\right)\right)$, where $y_{i,n}(x)$ are defined by formulas (3). Denote $\hat{e}_0=(0,1)$, $\hat{e}_n=(\cos\pi nx;\ 0)$, $n\in Z^{+}$.

Theorem 1. *The system $\{\hat{y}_{i,n}\}_{i=1,2;n\in Z^{+}}$ of eigenvectors of the operator L forms a basis in the space $L_{p,\nu}(0,1)\oplus C$ $1<p<\infty$, equivalent to the system $\hat{e}_0\cup\{\,\hat{e}_n\}_{n\in Z^{+}}$.*

Theorem 2. *If any function is excluded from the system $\{y_{i,n}\}_{i=1,2;\,n\in Z^{+}}^{\infty}$ of eigenfunctions of problem (1), (2), then the resulting system forms a basis in $L_{p,\nu}(0,1)$ equivalent to the system $\{\cos\pi nx\}_{n\in Z^{+}}$.*

Note that in the case when boundary conditions $y(0)=y(1)=0$, are taken instead of boundary conditions $y'(0)=y'(1)=0$, similar issues are studied in [4,5].

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On an ınverse problem for a parabolıc equatıon ın a domaın wıth movıng boundarıes

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The paper considers to investigate corrects of the inverse problem for finding the unknown right side, which depends on the time variable. It is considered the Neumann mixed boundary value problem on domain which the boundary depends on the time variable, an additional condition for finding the unknown function is given in the integral form. Theorems on the uniqueness and "conditional" stability of the solution and on the existence of a generalized solution of the considered inverse problem are proved.

Let $x = \gamma(t), 0 < a = \gamma(0) \leq \gamma(t) \leq \gamma(T) = b < \infty$, $0 < T = const$, $a, b = const$.

We consider the following inverse problem on determining a pair of functions $\{f(t), u(x,t)\}$:

$$u_t - u_{xx} = f(t)g(x), \;\; (x,t) \in D = (0, \gamma(t)) \times (0, T], \tag{1}$$

$$u(x,0) = \varphi(x), \quad x \in [0,a], \tag{2}$$

$$u_x(0,t) = \psi_0(t), \quad u_x(\gamma(t),t) = \psi_1(t), \;\; t \in [0,T], \tag{3}$$

$$\int_0^c u(x,t)dx = h(t), \quad t \in [0,T]. \tag{4}$$

where c is a constant such that $0 < c < a$, $g(x), \varphi(x,t), \psi_0(t), \psi_1(t), h(t), \gamma(t)$ are given functions.

We take the following assumptions for the data of problem (1)-(4):

1.1. $g(x) \in C^{\alpha}([0,b])$, $\int_0^c g(x)dx = g_0 \neq 0$, (without loss of the generality, for the prostate the sake of clarity, we take $g_0 = 1$)

1.2. $\varphi(x) \in C^{2+\alpha}[0,a]$, $\int_0^c \varphi(x)dx = h(0)$;

1.3. $\psi_0(t), \psi_1(t) \in C^{1+\alpha}[0,T]; \varphi'(0) = \psi_0(0)$; $\varphi'(a) = \psi_1(a)$

1.4. $h(t) \in C^{1+\alpha}[0,T]$, $t \in [0,T]$.

1.5. $\gamma(t) \in C^{1+\alpha}[0,T]$, $\gamma'(t) > 0$, $t \in [0,T], 0 < a = \gamma(0) \leq \gamma(t) \leq \gamma(T) = b < +\infty$.

Definition 1. *The pair of functions $\{f(t), u(x,t)\}$ is called the classical solutions of problem (1)-(4) :*

1. $f(t) \in C^{\alpha}[0,T]$;
2. $u(x,t) \in C^{2+\alpha,1+\alpha/2}(D) \bigcap C^{1+\alpha,(1+\alpha)/2}(\bar{D})$;
3. *the conditions (1)-(4) hold for these functions.*

Lemma 1. *Let us assume that, conditions 1.1-1.5 is satisfied. Then the problems (1)-(4) and (1),(2),(3),(5) have a classical solution* $\{f(t), u(x,t)\}$ *in the sense definition 1, then these functions are solution and takes (1)-(3) and*

$$f(t) = h'(t) - u_x(c,t) + \psi_0(t), \quad t \in [0,T], \tag{5}$$

are equivalent.

Define the following set:

$$K_\alpha = \{(f,u) | \, f(t) \in C^{\alpha}[0,T], \, |f(t)| \le c_1, \; t \in [0,T], \; u(x,t) \in C^{2+\alpha,\,1+\alpha/2}(\bar{D}),$$

$$|u|, |u_x|, |u_{xx}| \le c_2, \; (x,t) \in \bar{D}\},$$

where c_1, c_2 are some positive constants. Let us assume that the two input functions $\{g(\cdot), \varphi(\cdot), \psi_1(\cdot), \psi_2(\cdot), h(\cdot)\}$ and $\{\bar{g}(\cdot), \; \bar{\varphi}(\cdot), \bar{\psi}_1(\cdot), \bar{\psi}_2(\cdot), \bar{h}(\cdot)\}$ are given for Problem A. For brevity of the further exposition, Problem A with the second input set we will call Problem $\bar{A}$. Let $\{f(t), u(x,t)\}$ and $\{\bar{f}(t), \bar{u}(x,t)\}$ be solutions of problems A and $\bar{A}$, respectively.

Theorem 1. *Let the following conditions hold*

1) The functions $\{g(\cdot), \varphi(\cdot), \psi_1(\cdot), \psi_2(\cdot), h(\cdot)\}$ *and*

$\{\bar{g}(\cdot), \bar{\varphi}(\cdot), \; \bar{\psi}_1(\cdot), \bar{\psi}_2(\cdot), \bar{h}(\cdot)\}$ *satisfy conditions 1.1 -1.4, respectively;*

2) The functions $\gamma_1(t), \gamma_2(t)$ *satisfy condition 1.5;*

3) Solutions of problems A *and* $\bar{A}$ *exist in the sense of the definition 1 and they belong to the set* K_α.

Then there exists a $T^*\,(0 < T^* \le T)$, *such that for* $(x,t) \in [\gamma_1(t), \gamma_2(t)] \times [0, T^*]$ *the solution of problem (1)-(4) is unique, and the stability estimate*

$$\|u - \bar{u}\|_D^{(0)} + \left\|f - \bar{f}\right\|_T^{(0)} \le c_3 \left[\|g - \bar{g}\|_{[a,b]}^{(0)} + \right.$$

$$\left. + \|\varphi - \bar{\varphi}\|_{[a,b]}^{(2)} + \left\|\psi_1 - \bar{\psi}_1\right\|_T^{(0)} + \left\|\psi_2 - \bar{\psi}_2\right\|_T^{(0)} + \left\|h - \bar{h}\right\|_T^{(1)} \right],$$

is valid, where $c_3 > 0$ *depends on the data of problems* A *and* $\bar{A}$ *in the set* K_α.

Definition 2. $\{f(t), u(x,t)\}$*functions called a generalized solution of the inverse problem (1)-(4) if:*

1) $f(t) \in C[0,T]$;

2) $u(x,t) \in C^{1,0}\left(\bar{D}\right)$;

3) these functions satisfy the system of integral equations (5) and (6):

$$u(x,t) = F(x,t) + y(x,t) - 2\int\limits_0^t G(x,t-\tau)\rho_1(\tau)d\tau +$$

$$+2\int\limits_0^t G(x-\gamma(\tau),t-\tau)\rho_2(\tau)d\tau, \tag{6}$$

where $G(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right)$, $t > 0$ *is the fundamental solution of equation* $y_t - y_{xx} = 0$,

$$F(x,t) = \tilde{\varphi}(x) + \frac{2\gamma(t)x - x^2}{2\gamma(t)}\left[\psi_0(t) - \psi_0(0)\right] + \frac{x^2}{2\gamma(t)}\left[\psi_1(t) - \psi_1(0)\right]$$

$$y(x,t) = \int\limits_0^t \int\limits_{-\infty}^{+\infty} G(x-\xi,t-\tau)\left[f(\tau)\tilde{g}(\xi) + \tilde{F}_{\xi\xi}(\xi,\tau) - \tilde{F}_\tau(\xi,\tau)\right]d\xi d\tau,$$

$\rho_1(t)$ *and* $\rho_2(t)$ *are the solution of the systems of integral equations*

$$-y_x(0,t) = \rho_1(t) + 2\int\limits_0^t G_x(-\gamma(\tau), t-\tau)\rho_2(\tau)d\tau,$$
$$-y_x(\gamma(t),t) = \rho_2(t) - 2\int\limits_0^t G_x(\gamma(t)-\gamma(\tau), t-\tau)\rho_1(\tau)d\tau.$$

Theorem 2. *Let the initial data of problems (1)-(4) satisfy conditions 1.1-1.5.*

Then there exists a $T_1 \in (0,T]$ *such that in the region* $\overline{D}_1 = [0,\gamma(t)] \times [0,T_1]$ *the inverse problem has a solution in the sense of the definition 2.*

On absolute convergence of spectral expansion in eigenfunctions of a third-order differential operator

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On the interval $G=(0,1)$ consider the differential operator

$$Lu=u^3+q_1\left(x\right)u^{(2)}+q_2\left(x\right)u^{(1)}+q_3\left(x\right)u$$

with coefficients $q_l\left(x\right)\epsilon L_2\left(G\right),\ l=\overline{1,3.}$
İn the present work, we study the problems of obsolete and uniform convergence of orthogonal expansions of functions of the class $W_1^1\left(G\right)$ in the orthonormal eigenfunctions of a differential operator L.

By $D(G)$ we denote the class of functions absolutely continuous together with their derivatives of order ≤ 2 on the interval $\overline{G}=[0,1]$.

An eigenfunction of the operator L corresponding to an eigenvalue λ is understood as a function $u(x)\epsilon D(G)$ that is not identically zero and satisfies the equation $Lu+\lambda u=0$ almost everywhere in G (see [1]).

Let $\{u_k(x)\}_{k=1}^{\infty}$ be a complete orthonormal system in $L_2(G)$ consisting of eigenfunctions of the operator L, and let $\{\lambda_k\}_{k=1}^{\infty}$ be the corresponding system of eigenvalues, $Re\ \lambda_k=0$.

By $W^{1_1}\left(G\right)$ we denote the class of functions $f(x)$ absolutely continuous on the interval $\overline{G}=[0,1]$ for which $f^{'}\left(x\right)\epsilon L_1\left(G\right)$.

We write $\mu_k=\left(-i\lambda_k\right)^{1/3}$ if $Im\ \lambda_k\geq 0$; $\mu_k=\left(i\lambda_k\right)^{1/3}$ if $Im\ \lambda_k<0$,

and introduce a partial sum of the spectral expansion of the function $f(x)\epsilon W_1^1\left(G\right)$ with respect to the system $\{u_k(x)\}_{k=1}^{\infty}$:

$$\sigma_\nu\left(x,f\right)=\sum_{\mu_k\leq\nu}f_k u_k\left(x\right),\qquad \nu>0$$

where $f_k=(f,u_k)=\int_G f(x)\overline{u_k\left(x\right)}dx$.

In this work , we prove the following theorem.

Theorem. Assume that $q_l\left(x\right)\epsilon L_2\left(G\right),\ l=\overline{1,3}$; function $f(x)\epsilon W_1^1\left(G\right)$ and system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy the conditons

$$\left|f\left(x\right)\overline{u_k^{(2)}(x)}\Big|_0^1\right|\leq C(f)\mu_k^{\alpha}\|u_k\|_\infty,\ \ 0\leq\alpha<2\ ,\ \mu_k\geq 1$$

and
$$\sum_{k=2}^{\infty} k^{-1}\omega_1\left(f^{'}, k^{-1}\right) < \infty, \quad \sum_{k=2}^{\infty} k^{-1}\omega_1\left(q_1 f, k^{-1}\right) < \infty,$$
Where $\omega(\cdot, \delta)$ is the modulus of continuity on the spase $L_1(G)$.

Then the spectral expansion of the function $f(x)$ with respect to the system $\{u_k(x)\}_{k=1}^{\infty}$ converges absolutely and uniformly on the interval $\overline{G} = [0, 1]$.

The absolute an uniform convergence of the expansion of functions from the class $W_p^1(G)$, $p > 1$ was proved in [2].

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The behavior of solutions of nonlinear suspension bridge problem with time-varying delay as $t \to +\infty$.

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We consider the following mathematical model for the oscillations of the bridge with a strong delay

$$\begin{cases} \mathcal{L}_1(u_1,u_2) = g_1(t,x,u_1,u_2), \\ \mathcal{L}_2(u_1,u_2) = g_2(t,x,u_1,u_2), \end{cases} \tag{1}$$

where

$$\mathcal{L}_1(u_1,u_2) = u_{1tt} + (a_1(x)u_{1xx})_{xx} + b[u_1-u_2]_+ +$$

$$+\lambda_1 u_{1t} + \sum_{j=1}^{2} \mu_{1j} u_{jt}(x, t-\tau_{1j}(t)),$$

$$\mathcal{L}_2(u_1,u_2) = u_{2tt} + (a_2(x)u_{1x})_x - b[u_1-u_2]_+ +$$

$$+\lambda_2 u_{1t} + \sum_{j=1}^{2} \mu_{2j} u_{jt}(x, t-\tau_{2j}(t)),$$

$$g_1(t,x,u_1,u_2) = -\xi_1|u_1|^{p_1-1}u_1 - \xi|u_1|^{q-1}|u_2|^{q+1}u_1 + h_1(t,x),$$

$0 < x < l$, $t > 0$, $u_1 = u_1(x,t)$ is state function of the road bed and $u_2 = u_2(x,t)$ is that of the main cable; λ, ξ, λ_i, ξ_i, μ_{ij}, are real numbers $i,j = 1,2$, $[k]_+ = \max\{k,0\}$, $\tau_{ij}(t) > 0$, $i,j = 1,2$, are time-varying delays.

Let's define the following initial and boundary conditions for the system (1).

$$u_1(0,t) = u_{1xx}(0,t) = u_1(l,t) = u_{1xx}(l,t) = 0, \ t > 0, \tag{2}$$

$$u_2(0,t) = u_2(l,t) = 0, t > 0, \tag{3}$$

$$u_i(x,0) = u_{i0}(x), \quad u_{it}(x,0) = u_{i1}(x), x \in (0,l), \ i = 1,2, \tag{4}$$

$$u_{it}\left(x,t-\tau_{ij}\left(t\right)\right)=f_{ij}\left(x,t-\tau_{ij}\left(t\right)\right),x\in\left(0,l\right),t\in\left(0,\tau_{ij}\left(0\right)\right),i,j=1,2. \quad (5)$$

For investigating the problem (1)-(5), we introduce the following notations:

$$H^k(a,b)=\left\{y:y,y^{'},...,y^{(k)}\in L_2(a,b)\right\},$$

$$\widehat{H}^k(a,b)=\left\{y:y\in H^k(a,b),\quad y^{(2s)}\left(a\right)=y^{(2s)}\left(b\right)=0,\ s=0,1,...,\left[\frac{k-1}{2}\right]\right\},$$

where $[a]$ is the integer part of the number a. We will denote the space $\widehat{H}^k(0,l)$ as $\widehat{H}^k$.

Assume that X is some Banach space and $J\subset$ R is any domain. Let's denote by $C(J,\ X)$ the set of continuous functions with respect to the norm of X whose values are from X and defined in J. If J is closed, $C(J,\ X)$ is a Banach space with respect to the norm $\|u\|_{C(J;X)}=\max_{t\in J}\|u(t)\|_X$.

Denote by $C^m(J,\ X)$ the set of m -times continuously differentiable functions with respect to the norm of X, whose values are from the Banach space X.

We will study the problem (1)-(5) under the following conditions

$$a_i\left(\cdot\right)\in C\left[0,l\ \right],\quad a_i\left(x\right)\geq a_{i0}>0,\ x\in\left[0,l\ \right], \quad (6)$$

$$b\geq 0,\ \ \eta,\ \lambda_i,\eta_i,\mu_{ij}\in R,\quad i,j=1,2, \quad (7)$$

$$\tau_{ij}(.)\in W_\infty^2(0,+\infty), \quad (8)$$

$$\tau_{ij}\left(t\right)>0,\tau_{ij}^{'}\left(t\right)\leq d_{ij}<1,\quad 0\leq t<+\infty,i,j=1,2, \quad (9)$$

$$h_i(\cdot)\in W_2^1([0,+\infty);\ L_2(0,l)),\ i=1,2. \quad (10)$$

The theorem on the existence and uniqueness of the local solution of the problem (1)-(5) is proved.

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On solvability of boundary-value problem for fourth-order elliptic equation with operator coefficients

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Let H be a separable Hilbert space with the scalar product (x, y), $x, y \in H$, and A a positive-definite self-adjoint operator in H ($A = A^* \geq cE$, $c > 0$, E is the identity operator). As is known, the domain of definition of the operator A^γ ($\gamma > 0$) becomes a Hilbert space H_γ with respect to the scalar product $(x, y)_\gamma = (A^\gamma x, A^\gamma y)$, $x, y \in D(A^\gamma)$.

Denote by $L_2(\mathbb{R}_+; H)$ the Hilbert space of all vector-functions defined on $\mathbb{R}_+$ with values in H and the norm

$$\|f\|_{L_2(\mathbb{R}_+;H)} = \left(\int_0^{+\infty} \|f(t)\|_H^2 \, dt \right)^{1/2}.$$

Next, let $L(X, Y)$ denote the set of linear bounded operators acting from a Hilbert space X into another Hilbert space Y; $\sigma(\cdot)$ be the spectrum of the operator $(\cdot)$; henceforth, everywhere derivatives are understood in the sense of the theory of distributions in a Hilbert space [1].

Now we introduce the following sets:

$$W_2^4(\mathbb{R}_+; H) = \left\{ u(t) : \ u^{(4)}(t) \in L_2(\mathbb{R}_+; H), \ A^4 u(t) \in L_2(\mathbb{R}_+; H) \right\},$$

$$\begin{aligned} W_{2,T}^4(\mathbb{R}_+; H) \ = \ & \left\{ u(t) : \ u(t) \in W_2^4(\mathbb{R}_+; H), u'(0) = T u''(0), u'''(0) = 0, \right. \\ & \left. T \in L\left(H_{3/2}, H_{5/2}\right) \right\}. \end{aligned}$$

Each of these sets, equipped with the norm

$$\|u\|_{W_2^4(\mathbb{R}_+;H)} = \left(\left\|u^{(4)}\right\|_{L_2(\mathbb{R}_+;H)}^2 + \left\|A^4 u\right\|_{L_2(\mathbb{R}_+;H)}^2 \right)^{1/2},$$

becomes a Hilbert space [1, Ch. 1].

Let's consider, in the space H, the boundary-value problem

$$u^{(4)}(t) + A^4 u(t) = f(t), \quad t \in \mathbb{R}_+, \tag{1}$$

$$u'(0) = Tu''(0), \quad u'''(0) = 0, \tag{2}$$

where

$$A = A^* \geq cE, \ c > 0, \quad T \in L\left(H_{3/2}, H_{5/2}\right),$$
$$f(t) \in L_2\left(\mathbb{R}_+; H\right), \ u(t) \in W_2^4\left(\mathbb{R}_+; H\right).$$

Definition. *If the vector-function $u(t) \in W_2^4(\mathbb{R}_+; H)$ satisfies Eq. (1) almost everywhere in $\mathbb{R}_+$, and the boundary conditions (2) are satisfied in the following sense:*

$$\lim_{t\to 0} \|u'(t) - Tu''(t)\|_{H_{5/2}} = 0, \quad \mathop{lim}_{t\to 0} \|u'''(t)\|_{H_{1/2}} = 0,$$

then $u(t)$ will be called a regular solution of the boundary-value problem (1), (2).

The following theorem holds.

Theorem. *Let $C = A^{5/2} T A^{-3/2}$ and the point $-\sqrt{2} \notin \sigma(C)$. Then the boundary-value problem* (1), (2) *for any $f(t) \in L_2(\mathbb{R}_+; H)$ has a unique regular solution representable in the form*

$$u(t) = \frac{1}{4}\int_0^{+\infty} G(t,s)\left(A^{-3} f(s)\right) ds + e^{\omega_1 A t}\left[A^{-7/2}\left(E + \frac{1}{\sqrt{2}}C\right)^{-1} A^{7/2}\varphi\right] -$$

$$-\omega_3 e^{\omega_2 A t}\left[\omega_3 A^{-7/2}\left(E + \frac{1}{\sqrt{2}}C\right)^{-1} A^{7/2}\varphi + A^{-3}\psi\right],$$

where

$$G(t,s) = \begin{cases} -\frac{1}{\omega_1} e^{\omega_2 A(t-s)} - \frac{1}{\omega_2} e^{\omega_1 A(t-s)}, & t - s > 0, \\ \frac{1}{\omega_3} e^{\omega_4 A(t-s)} + \frac{1}{\omega_4} e^{\omega_3 A(t-s)}, & t - s < 0, \end{cases}$$

$$\varphi = \frac{\omega_2}{2} A^{-1}\left[T\xi - \eta - \omega_1 (T - \omega_1 A^{-1}) A^{-1}\psi\right],$$

$$\xi = \frac{1}{4}\int_0^{+\infty}\left(\omega_2 e^{-\omega_4 A s} + \omega_1 e^{-\omega_3 A s}\right)\left(A^{-1} f(s)\right) ds,$$

$$\eta = \frac{1}{4}\frac{\omega_4}{\omega_3}\int_0^{+\infty}\left(e^{-\omega_4 A s} - e^{-\omega_3 A s}\right)\left(A^{-2} f(s)\right) ds,$$

$$\psi = -\frac{1}{4}\int_0^{+\infty} \left(e^{-\omega_4 As} + e^{-\omega_3 As}\ \right) f(s)ds,$$

$$\omega_1 = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ \omega_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, \omega_3 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ \omega_4 = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$$

Note that similar issues for second-order elliptic-type equations were studied in [2].

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Problems of solvability of a boundary value problem with operator boundary conditions for an elliptic operator-differential equation of the second order

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Before turning to the topic of this paper, we briefly outline some of the results obtained in [1], which is close to this paper in terms of the formulation of the problem. In [1], in a UMD Banach space E, the questions of the solvability of the following boundary value problem for an elliptic differential-operator equation of the second order were studied:

$$u''(x) + Au(x) - \omega u(x) = f(x), \quad x \in (0,1) \tag{1}$$

$$\begin{array}{l} Bu'(0) + u(1) = f_1, \\ u(0) = f_2, \end{array} \tag{2}$$

where $\omega \geq \omega_0$ ($\omega_0 \geq 0$ fixed number) is the spectral parameter. With respect to the boundary value problem (1), (2), we have proved the following

Theorem 1. (see [1], Theorem 4.4) *Let the following conditions be satisfied:*

1) $A_{\omega_0} = A - \omega_0 I$ *is a linear closed operator in* E *with domain definition* $D(A)$, $[0,+\infty) \subset \rho(A_{\omega_0})$ ($\rho(A_{\omega_0})$ *is the resolvent set of the operator* A_{ω_0}) *and* $\sup\limits_{\lambda\geq 0} \|\lambda(A_{\omega_0} - \lambda I)^{-1}\|_{L(E)} < +\infty$;

2) *There are bounded imaginary powers of the operator* $-A$;

3) B *linear closed operator in* E *with domain definition* $D(B)$, $D(B) \supset D(Q_{\omega_0})$, *where* $Q_{\omega_0} = -(-A_{\omega_0})^{\frac{1}{2}}$, *for any* $\upsilon \in D(B)$ $A_{\omega_0}^{-1} B\upsilon = BA_{\omega_0}^{-1}\upsilon$;

4) $f_1, f_2 \in E$ *and* $f \in L_p((0,1);E)$.

Then there exists $\omega^* \geq \omega_0$ *such that, for any* $\omega \geq \omega^*$ *problem (1),(2) has a unique strong solution (i.e., solutions belonging to the space* $W_p^2((0,1);D(A),E)$*) if and only if*

$$f_2 \in (D(A),E)_{\frac{1}{2p},p} \subset D(Q),\ Q_\omega f_2 - \Gamma f \in D(B)$$

and

$$f_1 - B(Q_\omega f_2 - \Gamma f) \in (D(A),E)_{\frac{1}{2p},p},$$

where

$$\Gamma f = \int_0^1 e^{sQ} f(s) ds.$$

In this paper, in a separable Hilbert space H, we study the solvability of the following boundary value problem for an elliptic differential-operator equation of the second order:

$$-u''(x) + Au(x) + \lambda u(x) = f(x), \quad x \in (0,1), \tag{3}$$

$$\begin{array}{l} Bu'(0) + u(1) = f_1, \\ u(0) = f_2, \end{array} \tag{4}$$

where λ_0 is a sufficiently large number; A is a positive operator in H with domain definition $D(A)$; B is a linear closed operator in H with domain definition $D(B)$, which is subordinate to the operator $A^{\frac{1}{2}}$ in a certain sense and is commutative with the inverse operator A^{-1} on $D(B)$, i.e. for any $\upsilon \in D(B)$, the equality: $A^{-1}B\upsilon = BA^{-1}\upsilon$ holds.

As you can see, if in the boundary value problem (1),(2) for UMD the Banach space E we take the Hilbert space H, replace the operator A through $-A$, take a sufficiently large number ω for the positive spectral parameter λ_0, then the roles of the spectrum and the resolvent set will change places and we get the boundary value problem (3),(4).

In this work, in contrast, to work [1], it is specifically indicated: from which spaces it is necessary to take data $f,\ f_1,\ f_2$ in order for the problem (3),(4) to have a unique strong solution. In addition, for a strong solution of problems (3),(4) some estimate in space $L_p((0,1);H),\ 1 < p < \infty$, is obtained, which is not available for boundary value problems (1),(2)

Theorem 2. *Let the following conditions be satisfied:*

1) *A is a positive operator in H;*

2) *The linear closed operator B is subordinate to the operator $A^{\frac{1}{2}}$ in H;*

3) *Operator B is commutative with operator A^{-1}.*

Then for $f \in L_p((0,1);H(A)),\ f_1 \in (H(A);H)_{\frac{1}{2p},p},\ f_2 \in (H(A^2);H)_{\frac{1}{4p},p}$ and for sufficiently large $\lambda_0 > 0$ the problem (3),(4) has a unique strong solution $u \in W_p^2((0,1);H(A),H)$ and the solution satisfies the following noncoer-

cive estimate

$$\|u\|_{W_p^2((0,1);H(A),H)} \leq C\Big(\|f\|_{L_p((0,1);H(A))} + \|f_1\|_{(H(A),H)_{\frac{1}{2p},p}} + \|f_2\|_{(H(A^2),H)_{\frac{1}{4p},p}}\Big)$$

Problems of solvability of boundary value problems for second-order elliptic operator-differential equations with a complex parameter in which the boundary conditions contain, generally speaking, a linear unbounded operator studied in [2].

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Discrete Hilbert transform in discrete Hölder spaces

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Let the function f be defined on the real axis and $\alpha \in (0,1]$. If there exists a number $M > 0$ such that for any $x, y \in R$

$$|f(x) - f(y)| \le M \cdot |x-y|^{\alpha} \tag{1}$$

and for any $x, y \in R \backslash \{0\}$

$$|f(x) - f(y)| \le M \cdot \left| \frac{1}{x} - \frac{1}{y} \right|^{\alpha}, \tag{2}$$

then the function f is said to be Holder continuous with exponent α in the real axis (see [5]). The class of Holder continuous functions with exponent α on the real axis with the norm

$$\|f\|_{\alpha} = \max_{x \in R} |f(x)| + \sup_{x,y \in R, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^{\alpha}} + \sup_{x,y \in R \backslash \{0\}, x \neq y} \frac{|f(x) - f(y)|}{|1/x - 1/y|^{\alpha}}$$

forms a Banach space and is denoted by $H_{\alpha}(R)$.

It follows from (1) and (2) that for any $f \in H_{\alpha}(R)$ there exist $f(\infty) = \lim_{x \to \pm\infty} f(x)$ and for any $x \neq 0$

$$|f(x) - f(\infty)| \le \frac{\|f\|_{\alpha}}{|x|^{\alpha}}.$$

Denote

$$H_{\alpha}^{0}(R) = \{f \in H_{\alpha}(R) : f(\infty) = 0\} \subset H_{\alpha}(R).$$

The Hilbert transform of a function $f \in H_{\alpha}^{0}(R)$, $\alpha \in (0,1]$ is defined as the Cauchy principle value integral

$$(Hf)(t) = \frac{1}{\pi} \int_{R} \frac{f(\tau)}{t-\tau} d\tau \equiv \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int_{R \backslash (t-\varepsilon, t+\varepsilon)} \frac{f(\tau)}{t-\tau} d\tau, \quad t \in R.$$

It is well known (see [5]) that the Hilbert transform of the function $f \in H_\alpha^0(R)$, $\alpha \in (0,1]$ exists for any $t \in R$. In case $\alpha \in (0,1)$, the Hilbert transform is a bounded map in the space $H_\alpha^0(R)$.

For a numerical sequence $b = \{b_n\}_{n\in Z}$ the sequence $H(b) = \{(Hb)_n\}_{n\in Z}$, where

$$(Hb)_n = \sum_{m\in Z, m\neq n} \frac{b_m}{n-m}, n \in Z,$$

is called the discrete Hilbert transform of the sequence $b = \{b_n\}_{n\in Z}$.

The Hilbert transform has been well studied on classical function spaces Holder, Lebesgue, Morrey, Campanato, Besov, Lorentz, Sobolev, etc. But its discrete version, which also has numerous applications, has not been fully studied in discrete analogues of these spaces. In [1, 2, 3, 4, 6, 7] were studied the properties of the discrete Hilbert transform in the discrete Lebesgue and Morrey spaces. In this paper we discuss the discrete Hilbert transform on discrete Holder spaces and obtain its boundedness on these spaces.

Let $b = \{b_n\}_{n\in Z}$ be a numerical sequence and $\alpha > 0$. If there exists a number $M > 0$ such that for any $m, n \in Z\backslash\{0\}$

$$|b_m - b_n| \leq M \cdot \left| \frac{1}{m} - \frac{1}{n} \right|^\alpha, \tag{3}$$

then the sequence $b = \{b_n\}_{n\in Z}$ is called a sequence satisfying the Hölder condition with exponent α. The class of sequences satisfying the Hölder condition with exponent α with norm

$$\|b\|_\alpha = \max_{n\in Z} |b_n| + \sup_{m,n\in Z\backslash\{0\}, m\neq n} \frac{|b_m - b_n|}{|1/m - 1/n|^\alpha}$$

forms a Banach space and is denoted by $h_\alpha(Z)$.

It follows from (3) that for any $b \in h_\alpha(Z)$ there exist $b_\infty = \lim_{n\to\pm\infty} b_n$ and for any $m \neq 0$

$$|b_m - b_\infty| \leq \frac{\|b\|_\alpha}{|m|^\alpha}.$$

Denote

$$h_\alpha^0(Z) = \{b \in h_\alpha(R) : b(\infty) = 0\} \subset h_\alpha(Z).$$

Note that in case $\alpha > 1$ the space $h_\alpha(Z)$ do not consist only of stationary sequences. Therefore, in this case, the discrete Hölder transform is also of interest.

Theorem 1. *For any $0 < \alpha < 1$ discrete Hilbert transform is a bounded map in the space $h^0_\alpha(Z)$, that is, there exists a constant $c_\alpha > 0$ depending only on $\alpha \in (0,1)$, such that for any $b \in h^0_\alpha(Z)$*

$$\|Hb\|_\alpha \leq c_\alpha \cdot \|b\|_\alpha.$$

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Application of Jacobian polynomials

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Construction of the Legendre, Chebyshev system of polynomials was given and some terms were determined in the previous works [2].

Let us consider Jacobian polynomials, one of the classic polynomials. For that, we will use an auxiliary lemma and theorem.

Lemma. *The n –th degree polynominal $Q_n(x)$ of the appropriate system of polynomials $F_0(x)$, $F_2(x), F_3(x), \ldots\ldots, F_n(x)$ can be uniquely shown as follows*

$$Q_n(x) = a_0F_0(x) + a_1F_1(x) + a_2F_2(x) + \cdots + a_nF_n(x).$$

Theorem. *There exists a unique system of polynomials $\{P_n(x)\}$ with positive higher degree coefficient and satisfying the orthogonality condition for any weight function $h(x)$ [1].*

We are given the weight function $h(x) = (1-x)^{\alpha}(1+x)^{\beta}\ x \in (-1,1)$, $\alpha > -1$, $\beta > -1$.

$P_n(x, \propto, \beta) = \frac{(-1)^n}{n!\cdot 2^n}(1-x)^{-\propto}(1+x)^{-\beta}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]^{(n)}$ is said to be a standarized Jacobian polynomial.

We can prove that, using the Euler integrals, according to the Leibnits formula,

$$P_n(x, \propto, \beta) = \frac{1}{n!\cdot 2^n}\frac{\Gamma(\alpha+\beta+2n+1)}{\Gamma(\alpha+\beta+n+1)}\cdot x^n +$$

$$+\frac{n(\alpha-\beta)}{n!\cdot 2^n}\frac{\Gamma(\alpha+\beta+2n)}{\Gamma(\alpha+\beta+n+1)}x^{n-1} + \cdots + 1$$

is an n-th order polynomial. It is clear that a higher degree unit Jacobian polynomial is as follows

$$\tilde{P}_n(x, \propto, \beta = (-1)^n \frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)(1-x)^{\alpha}\cdot(1+x)^{\beta}}\cdot\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]^{(n)}.$$

It is known that with respect to the weight function $h(x)$ the Jacobian polynomial is orthogonal in $(-1,1)$, i.e. for $m < n$ we see that $\{P_n(x, \propto, \beta)\}$ is orthonormal

$$J_{mn} = \int_{-1}^{1} (x-1)^{-\propto}(1+x)^{-\beta} P_m(x,\ \propto,\ \beta) P_n(\alpha,\ \alpha,\ \beta)dx = 0.$$

As a result, we determine the Jacobian polynomial.

$$\overline{P}_n(x, \alpha, \beta) = \sqrt{\frac{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1}} P_n(x, \alpha, \beta).$$

By means of the new method, using the existence and uniqueness condition let us consider the construction of the system of Jacobian polynomials:

$$\overline{P}_0(x, \alpha, \beta,) = \mu_0(\alpha, \beta,) = \mu_0$$

$$\int_{-1}^{1} \overline{P}_0^2(x, \alpha, \beta) h(x) dx = 1$$

$$\mu_0^2 \int_{-1}^{1} (1-x)^{\alpha} \cdot (1+x)^{\beta} dx = 1$$

Replace $x = 2t-1, x = -1, t = 0, x = 1, t = 1, dx = 2dt$

$$\mu_0^2 \int_{0}^{1} (1-2t+1)^{\alpha}(1+2t-1)^{\beta} \cdot 2dt = 1$$

$$\mu_0^2 \int_{0}^{1} 2^{\alpha}(1-t)^{\alpha} \cdot 2^{\beta} t^{\beta} \cdot 2dt = 1$$

$$\mu_0^2 \cdot 2^{\alpha+\beta+1} \int_{0}^{1} (1-t)^{\alpha} \cdot t^{\beta} dt = 1$$

$$\mu_0^2 \cdot 2^{\alpha+\beta+1} B(\alpha+1) B(\beta+1) = 1$$

Using the relation between beta and gamma

$$\mu_0^2 \cdot 2^{\alpha+\beta+1} \cdot \frac{2^{\alpha+\beta+2n+1}\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} = 1$$

$$\mu_0 \cdot \frac{\sqrt{\Gamma(\alpha+\beta+2)}}{\sqrt{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}} = 1$$

$$\mu_0 = \sqrt{\frac{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}}$$

$$\overline{P}_0\,(x,\alpha,\beta) = \sqrt{\frac{(\alpha+\beta+1)\Gamma(\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}}$$

Determine, $\overline{P}_1\,(x,\alpha,\beta)$. Using the orthonormality condition,

$$\begin{cases} \int_{-1}^{1} h\,(x)\, P_0\,(x,\alpha,\beta)\cdot P_1\,(x,\alpha,\beta)\, dx = 0 \\ \int_{-1}^{1}\ h\,(x)\, P_1^2\ \ (x,\ \ \alpha,\ \ \beta\ )\ \ dx = 1 \end{cases}$$

Using the appropriate lemma, $\overline{P}_1\,(x,\alpha,\beta) = C_0 + C_1 x$

$$\begin{cases} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\overline{P}_0\,(x,\alpha,\beta)\,(C_0 + C_1 x)\, dx = 0 \\ \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}(C_0 + C_1 x)^2 dx = 1 \end{cases}$$

As a result ,

$$C_0 = \sqrt{\frac{(\alpha+\beta+3)\,\Gamma\,(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma\,(\alpha+2)\,\Gamma\,(\beta+2)}}\cdot\frac{1}{2}(\alpha-\beta)$$

$$C_1 = \sqrt{\frac{(\alpha+\beta+3)\,\Gamma\,(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma\,(\alpha+2)\,\Gamma\,(\beta+2)}}\cdot\frac{1}{2}(\alpha+\beta+2)$$

So, we determined the polynomial $\overline{P}_1\,(x,\alpha,\beta) = c_0 + c_1\ x$.

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On limit behavior of the Markov random walks describes by the generalization of the autoregressive process of order one $(AR(1))$

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Let (Ω, F, P) be a probability space, and $\{\xi_n,\ n \geq 1\}$ be a sequence of independent random variables with $E\xi_n = 0$ and $E\xi_n = \sigma_n^2$. Define the sequence of random variables $\{X_n\}$ by

$$X_n = \theta_0 X_{n-1} + \xi_n \tag{1}$$

for some fixed number $\theta_0 \in (-\infty, \infty)$, where initial value X_0 is independent of $\{\xi_n\}$.

In the case of independent and identically distributed (i.i.d) random variables ξ_n the process is called an autoreqressive process of order one $(AR(1))$.

As noted in [3], the statistical estimate for the parameter θ_0 based on observations $X_0, X_1, ..., X_n$ has the form

$$\theta_n = \frac{T_n}{S_n},$$

where

$$T_n = \sum_{k=1}^{n} X_k X_{k-1} \quad \text{and} \quad S_n = \sum_{k=1}^{n} X_{k-1}^2, n \geq 1.$$

Note that the statistical estimate θ_n was obtained using the least squares method [1]. The sequences T_n, S_n and $\theta_n, n \geq 1$ are Markov random walks and play an important role in the theory of nonlinear renewal.

Limit theorems for these Markov random walks were studied in under various assumptions about the innovation ξ_n and the parameter θ_0 of the process $AR(1)$.

These limit theorems allow us to study a number of boundary value problems related to the intersection of a linear and nonlinear boundary by random walks T_n, S_n and $\theta_n, n \geq 1$.

Some linear boundaries of the problem for Markov random walks are studied in [1- 3].

We know that [3] the least-squares estimator θ_n for θ_0 (1) gives

$$\theta_n = \frac{\sum\limits_{i=1}^{n} \frac{X_i X_{i-1}}{\sigma_i^2}}{\sum\limits_{i=1}^{n} \left(\frac{X_{i-1}}{\sigma_i}\right)^2}. \tag{2}$$

In [3] was shown that under conditions

$$\sup_i \frac{\sigma_{i+1}^2}{\sigma_i^2} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} E\left(\frac{\xi_n^2}{\sigma_n^2} \wedge 1\right) = \infty, \tag{3}$$

where $(a \wedge 1) = \min(a, 1)$ convergence almost surely $\theta_n \overset{a.s.}{\to} \theta_0$ as $n \to \infty$ is true.

Set

$$A_n = \sum_{i=1}^{n} \frac{X_i X_{i-1}}{\sigma_i^2}, \quad M_n = \sum_{i=1}^{n} \frac{\xi_i X_{i-1}}{\sigma_i^2}, \quad B_n = \sum_{i=1}^{n} \left(\frac{X_{i-1}}{\sigma_i}\right)^2.$$

Then we have from (2)

$$\theta_n = \frac{A_n}{B_n} = \theta_0 + \frac{M_n}{B_n}.$$

It follows that convergence almost surely

$$\frac{M_n}{B_n} \overset{a.s.}{\to} 0 \quad \text{of} \quad n \to \infty \tag{4}$$

is the necessary and sufficient condition for $\theta_n \overset{a.s.}{\to} \theta_0$ of $n \to \infty$.

In was shown that conditions (3) are sufficient for (4).

Note that in work [3] for the case of (i.i.d) random variables ξ_n with $E\xi_1 = 0$ and $E\xi_1^2 = 1$ was shown that if $|\theta_0| < 1$ and $EX_0^2 < \infty$ as $n \to \infty$

$$\frac{A_n}{n} \overset{a.s.}{\to} \frac{\theta_0}{1 - \theta_0^2}, \quad \frac{B_n}{n} \overset{a.s.}{\to} \frac{1}{1 - \theta_0^2},$$

$$D_n = \frac{A_n^2}{B_n} \xrightarrow{a.s.} \frac{\theta_0^2}{1-\theta_0^2}. \tag{5}$$

Furthermore, in [3] for the case of (i.i.d.) random variables ξ_n with $E\xi_1 = 0$ and $E\xi_1^2 < \infty$ is obtained the following result: if $|\theta_0| < 1$ and $EX_0^2 < \infty$, then

$$\lim_{n\to\infty} P\left(\sqrt{n}\left(\theta_n - \theta_i\right) \le x\right) = \Phi\left(\frac{x}{\sqrt{1-\theta_0^2}}\right), \quad x \in R = (-\infty, \infty), \tag{6}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-y^2/2} dy$.

In special case, the following theorem was obtained.

Theorem. *Let $\{\xi_n,\ n \ge 1\}$ be a sequence of independent random variables with $E\xi_n = 0$, $E\xi_n^2 = 1$. Suppose that $\sum\limits_{n=1}^{\infty} E\left(\xi_n^2 \wedge 1\right) = \infty$ and $|\theta_0| < 1$, $EX_0^2 < \infty$.*

$$\frac{B_n}{n} \xrightarrow{a.s.} \frac{1}{1-\theta_0^2},$$

$$\sqrt{n}\left(\frac{B_n}{n} - \frac{1}{1-\theta_0^2}\right) \xrightarrow{d} N(0, \alpha_1),$$

where $\alpha_1 = \frac{1}{|\theta_0|^2(1-\theta_0)^2}$.

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Basicity of exponent system in the rearrangement invariant space

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In this work rearrangement invariant space is considered and the norm generated by it of the Hardy classes of analytic functions inside and outside the unit ball, respectively.

We will need some concepts and facts from the theory of Banach function spaces (see, e.g., [1,2]). Let $(M;\mu)$ be a measure space, and $\mathscr{M}^+$ be a cone of μ-measurable functions on M whose values lie in $[0,+\infty]$.

Let $\mathscr{M}$ denote the collection of all extended scalar-valued (real or complex) μ-measurable functions and $\mathscr{M}_0 \subset \mathscr{M}$ denote the subclass of functions that are finite μ-a.e..

If equipped with the norm $\|f\|_X = \rho\left(|f|\right)$, X is called a Banach function space. Let

$$\rho'\left(g\right) = \sup\left\{\int_\gamma f\left(\tau\right) g\left(\tau\right)|dt| : f \in \mathscr{M}^+ ; \rho\left(f\right) \leq 1\right\}, \forall g \in \mathscr{M}^+.$$

A space

$$X' = \left\{g \in \mathscr{M} : \rho'\left(|g|\right) < +\infty\right\},$$

is called an associate space (Kothe dual) of X.

We will consider the case $M = (-\pi;\pi]$ (or $M = \gamma$), $\mu = m$ is Lebesgue measure. In addition, we will identify $(-\pi;\pi]$ and γ.

The functions $f; g \in \mathscr{M}_0$ are called equimeasurable if

$$\left|\left\{\tau \in M : |f\left(\tau\right)| > \lambda\right\}\right| = \left|\left\{\tau \in M : |g\left(\tau\right)| > \lambda\right\}\right|, \forall \lambda \geq 0.$$

Banach function norm $\rho : \mathscr{M}^+ \to [0,\infty]$ is called rearrangement invariant if for arbitrary equimeasurable functions $f; g \in \mathscr{M}_0^+$ the relation $\rho\left(f\right) = \rho\left(g\right)$ holds. In this case, Banach function space X with the norm $\|\cdot\|_X = \rho\left(|\cdot|\right)$ is said to be rearrangement invariant function space (r.i.s. for short). Classical Lebesgue, Orlicz, Lorentz, Lorentz-Orlicz spaces are r.i.s.

Let α_X and β_X be upper and lower Boyd indices for the space X (for Boyd indices, we refer the readers to, e.g., [1,2]).

Let us state the following concept.

Definition 1. *Let X be a Banach function space. The closure of the set of simple functions $\mathscr{M}_s$ in X is denoted by X_b.*

The following theorem is proved.

Theorem 1. *Let X be an r.i.s. on γ. Then the system of exponents $\{e^{int}\}_{n\in Z}$ forms a basis for X_b if and only if the Boyd indices of X satisfy $0 < \alpha_X; \beta_X < 1$.*

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The boundedness of maximal operator in the weighted Morrey spaces, associated with the Laplace-Dunkl differential operator

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The well-known Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ introduced in relation to the study of partial differential equations, and presented in various books, were widely investigated during the last decades, including the study of classical operators of harmonic analysis-maximal, singular and potential operators-in these spaces.

In this paper, we deal with the multidimensional case and study the boundedness of the maximal operator in the weighted Morrey spaces, associated with the Laplace-Dunkl differential operator.

Let $\alpha > -1/2$ be a fixed number and μ_α be the weighted Lebesgue measure on $\mathbb{R}^n$, given by $d\mu_\alpha(x) := \left(2^{\alpha+1}\Gamma(\alpha+1)\right)^{-1} |x|^{2\alpha+1}\, dx$.

The weight function ω belongs to the class $A_{p,\alpha}(\mathbb{R}^n)$ for $1 \le p < \infty$ if the following statement

$$\sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\mu_\alpha B(x,r)} \int\limits_{B(x,r)} \omega(y) d\mu_\alpha(y) \left(\frac{1}{\mu_\alpha B(x,r)} \int\limits_{B(x,r)} \omega^{-\frac{1}{p-1}}(y) d\mu_\alpha(y) \right)^{p-1}$$

is finite.

Definition 1. *Let $1 \le p < \infty$, $0 \le \lambda \le n + 2\alpha + 1$. We denote by $\mathcal{M}_{p,\lambda,\alpha}(\mathbb{R}^n)$ Dunkl-type Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norm*

$$\|f\|_{\mathcal{M}_{p,\lambda,\alpha}} = \sup_{t>0, x\in\mathbb{R}^n} \left(t^{-\lambda} \int\limits_{B_t} [\tau_x |f(y)|]^p d\mu_\alpha(y) \right)^{1/p}.$$

The operators τ_x, $x \in \mathbb{R}$ are called Dunkl translation operators on $\mathbb{R}$ and it can be expressed in the following form

$$\tau_x f(y) = c_\alpha \int_0^\pi f_e\left((x_1, y_1)_\theta, x' - y'\right)\, h_1(x_1, y_1, \theta)(\sin\theta)^{2\alpha}\, d\theta$$

$$+c_\alpha \int_0^\pi f_o\left((x_1,y_1)_\theta, x'-y'\right)\, h_2(x_1,y_1,\theta)(\sin\theta)^{2\alpha}\, d\theta,$$

where $(x_1,y_1)_\theta = \sqrt{x_1^2+y_1^2-2|x_1y_1|\cos\theta}$, $f = f_e + f_o$, f_o and f_e being respectively the odd and the even parts of f, with

$$c_\alpha \equiv \left(\int_0^\pi (\sin\theta)^{2\alpha}\, d\theta\right)^{-1} = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)}, \quad h_1(x_1,y_1,\theta) = 1 - sgn(x_1y_1)\cos\theta$$

and

$$h_2(x_1,y_1,\theta) = \begin{cases} \frac{(x_1+y_1)\,[1-sgn(x_1y_1)\cos\theta]}{(x_1,y_1)_\theta}, & \text{if } \ x_1y_1 \neq 0, \\ 0, & \text{if } \ x_1y_1 = 0. \end{cases}$$

Now we define the Dunkl-type maximal operator by

$$M_\alpha f(x) = \sup_{r>0} \left(\mu_\alpha B_r\right)^{-1} \int_{B_r} \tau_x |f|(y)\, d\mu_\alpha(y).$$

Theorem 1. *Let $1 < p < \infty$, $0 \le \lambda < n+2\alpha+1$, then the boundedness M_α from $\mathcal{M}_{p,\lambda,\alpha}(\mathbb{R}^n)$ to $\mathcal{M}_{p,\lambda,\alpha}(\mathbb{R}^n)$.*

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On boundedness of multidimensional Hausdorff operators in weighted Lebesgue spaces

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In this abstract, we establish two-weight norm inequalities for multidimensional Hausdorff operators in weighted Lebesgue spaces. In particular, we establish necessary and sufficient conditions for the boundedness of special-type multidimensional Hausdorff operators in weighted Lebesgue spaces for monotone radial weight functions. Also, we get similar results for important operators of harmonic analysis which are special cases of the multidimensional Hausdorff operators.

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Global bifurcation from infinity in nonlinear eigenvalue problems for ordinary differential equations of fourth order with a spectral parameter in the boundary condition

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This note is devoted to the study of the following nonlinear eigenvalue problem

$$(py'')'' - (qy')' = \lambda ry + h(x, y, y', y'', y''', \lambda),\ x \in (0, l), \tag{1}$$

$$y'(0) \cos\alpha - (py'')(0) \sin\alpha = 0, \tag{2}$$

$$y(0) \cos\beta + Ty(0) \sin\beta = 0, \tag{3}$$

$$y'(l) \cos\gamma + (py'')(l) \sin\gamma = 0, \tag{4}$$

$$(a\lambda + b)y(l) - (c\lambda + d)Ty(l) = 0, \tag{5}$$

where $\lambda \in \mathbb{R}$ is an eigenvalue parameter, $Ty \equiv (py'')' - qy'$, $p \in C^2([0, l]; \mathbb{R}^+)$, $q \in C^1([0, l]; \mathbb{R}_0^+)$, $r \in C([0, l]; \mathbb{R}^+)$, $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{R}_0^+ = [0, +\infty)$, α, β, γ and a, b, c, d are real constants such that $\alpha, \beta, \gamma \in [0, \pi/2]$ and $\sigma = bc - ad > 0$. Moreover, the nonlinear term h has the form $f + g$, where f and g are real-valued continuous functions on $[0, l] \times \mathbb{R}^5$ that satisfy the following conditions: there exist positive constants M and K such that

$$\begin{gathered} \left|\frac{f(x,y,s,v,w,\lambda)}{y}\right| \leq M,\ x \in [0, l],\ (y, s, v, w) \in \mathbb{R}^4,\ y \neq 0, \\ |y| + |s| + |v| + |w| \geq K,\ \lambda \in \mathbb{R}; \end{gathered} \tag{6}$$

for any fixed bounded interval $\Lambda \subset \mathbb{R}$

$$g(x, y, v, \vartheta, w, \lambda) = o\left(|y| + |v| + |\vartheta| + |w|\right) \text{ as } |y| + |v| + |\vartheta| + |w| \to \infty, \tag{7}$$

uniformly in $(x, \lambda) \in [0, l] \times \Lambda$.

Let BC_0 is the set of functions satisfying boundary conditions (2)-(4).

To preserve the nodal properties in [1] the authors constructed sets S_k^ν, $k \in \mathbb{N}$, $\nu \in \{+\,,\,-\}$ of functions in Banach space $E = C^3[0,l] \cap BC_0$ with the usual norm of $C^3[0,l]$ that have the oscillatory properties of eigenfunctions of the linear eigenvalue problem (1)-(5) with $h \equiv 0$ and their derivatives. Moreover, in [1] in the case when g satisfies $o(|y| + |y'| + |y''| + |y'''|)$ condition it was proved the existence of global continua of nontrivial solutions of problem (1)-(5) bifurcating from zero and contained in classes $\mathbb{R} \times S_k^\nu$, $k \in \mathbb{N}$, $\nu \in \{+\,,\,-\}$.

The global bifurcation from infinity in problem (1)-(5) with $f \equiv 0$ was studied in [2], where it was shown the existence of global continua of solutions which have properties similar to those of the continua found in Rabonowitz' well-known global asymptotic bifurcation theorem.

Let

$$\mathcal{I}_k = [\lambda_k - M/r_0, \lambda_k + M/r_0],$$

where λ_k is the kth eigenvalue of problem (1)-(5) with $h \equiv 0$, which is simple [1], and $r_0 = \min\limits_{x \in [0,l]} r(x)$.

The main result of this paper is the following theorem

Theorem 1. *For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ there exists a connected and closed subset $\mathcal{D}_k^\nu$ of the set of solutions of problem (1)-(5) and a neighborhood $\mathcal{Q}_k$ of $\mathcal{I}_k \times \{\infty\}$ such that (i) $\mathcal{I}_k \times \{\infty\} \subset \mathcal{D}_k^\nu$; (ii) $(\mathcal{D}_k^\nu \backslash (\mathcal{I}_k \times \{\infty\})) \cap \mathcal{Q}_k \subset \mathbb{R} \times S_k^\nu$; (iii) either $\mathcal{D}_k^\nu$ meets $\mathcal{I}_{k'} \times \{\infty\}$ with respect to $\mathbb{R} \times S_{k'}^{\nu'}$ for some $(k', \nu') \neq (k, \nu)$ or $\mathcal{D}_k^\nu$ meets $\mathbb{R} \times \{0\}$ for some $\lambda \in \mathbb{R}$ or the projection of $\mathcal{D}_k^\nu$ on $\mathbb{R} \times \{0\}$ is unbounded.*

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On a spectral problem for the Dirac system with boundary conditions depending on the spectral parameter

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We consider the one-dimensional Dirac system

$$v' - \{\lambda + p\,(x)\}\,u = 0,\;\; u' + \{\lambda + r(x)\}\,v = 0,\;\; x \in (0, \pi), \tag{1}$$

with the boundary conditions

$$(\lambda \cos\alpha + a_0)\,v(0) + (\lambda \sin\alpha + b_0)\,u\,(0) = 0, \tag{2}$$

$$(\lambda \cos\beta + a_\pi)\,v(\pi) + (\lambda \sin\beta + b_0)\,u\,(\pi) = 0, \tag{3}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $p\,(x)$ and $r(x)$ are continuous functions on the interval $[0,\ \pi]$, α, β, a_0, b_0, a_1 and b_1 are real constants such that $0 \leq \alpha,\ \beta < \pi$ and

$$\sigma_0 = a_0 \sin\alpha - b_0 \cos\alpha < 0,\;\; \sigma_1 = a_1 \sin\beta - b_1 \cos\beta > 0.$$

If $p\,(x) = V(x) + m,\ r(x) = V(x) - m$, where $V(x)$ is a potential function, and m is the mass of a particle, then the system (1) is known in relativistic quantum theory a stationary one-dimensional Dirac system or the first canonical form of the Dirac system [1].

The oscillatory properties of components of all eigenvector-functions of the problem (1)-(3) were studied in [1] for $a_0 = b_0 = a_1 = b_1 = 0$, and in [2] for $a_0 = b_0 = 0$. Moreover, earlier in the papers [3] and [4], the oscillation properties of eigenvector-functions of the Dirac system (1)-(3) with $-\frac{\pi}{2} \leq \alpha,\ \beta < \frac{\pi}{2}$ were studied. It should be noted that in these papers the authors failed to establish the exact number of zeros of components of the eigenvector-function corresponding to the kth eigenvalue (although for sufficiently large $|k|$, $k \in \mathbb{N}$).

Lemma 1. *For each fixed $\lambda \in \mathbb{C}$ there exists a unique solution $w(x,\lambda) = \begin{pmatrix} u(x,\lambda) \\ v(x,\lambda) \end{pmatrix}$ of system (1) which satisfy the initial condition*

$$u(0,\lambda) = \lambda\cos\alpha + a_0, \quad v(0,\lambda) = -(\lambda\sin\alpha + b_0). \tag{4}$$

For each fixed $x \in [0, \pi]$ the functions $u(x,\lambda)$ and $v(x,\lambda)$ are entire functions of λ.

Let

$$\varphi(\lambda) = \frac{\lambda\cos\alpha + a_0}{\lambda\sin\alpha + b_0}, \ \psi(\lambda) = \frac{\lambda\cos\beta + a_1}{\lambda\sin\beta + b_1}.$$

We define a continuous function $\tau(\lambda)$, $\chi(\lambda)$ on $\mathbb{R}$ as follows:

$$\begin{gathered} \tau(\lambda) = \pi - \cot^{-1}\varphi(\lambda) \ \text{ if } \alpha = 0 \text{ or } \alpha \neq 0 \text{ and } \lambda \in \left(-\tfrac{b_0}{\sin\alpha}, \ +\infty\right), \\ \tau(\lambda) = -\cot^{-1}\varphi(\lambda) \ \text{ if } \alpha \neq 0 \text{ and } \lambda \in \left(-\infty, \ -\tfrac{b_0}{\sin\alpha}\right]. \end{gathered} \tag{5}$$

$$\begin{gathered} \chi(\lambda) = \pi - \cot^{-1}\psi(\lambda) \text{ if } \beta = 0 \text{ or } \beta \neq 0 \text{ and } \lambda \in \left(-\infty, \ -\tfrac{b_1}{\sin\beta}\right], \\ \chi(\lambda) = -\cot^{-1}\psi(\lambda) \ \text{ if } \beta \neq 0 \text{ and } \lambda \in \left(-\tfrac{b_1}{\sin\beta}, \ +\infty\right). \end{gathered} \tag{6}$$

Then we have

$$\cot\varphi(\lambda) = -\varphi(\lambda) = \frac{\lambda\cos\alpha + a_0}{\lambda\sin\alpha + b_0}, \ \varphi\left(-\frac{b_0}{\sin\alpha}\right) = 0,$$

$$\cot\chi(\lambda) = -\psi(\lambda) = \frac{\lambda\cos\beta + a_1}{\lambda\sin\beta + b_1}, \ \psi\left(-\frac{b_1}{\sin\beta}\right) = 0.$$

For the study the oscillation properties of eigenvector-functions of problem (1)-(3), we use the Prüfer angular function

$$\theta(x,\lambda) = \tan^{-1}(v(x,\lambda)/u(x,\lambda)),$$

or more precisely,

$$\theta(x,\lambda) = \arg\{u(x,\lambda) + iv(x,\lambda)\}.$$

Since u, v are functions of variables x and λ, so also is θ, and we determine initially

$$\theta(0, \lambda) = \tau(\lambda),$$

in view of (3).

The main result of the present work is the following theorem.

Theorem 1. *The eigenvalues λ_k, $k \in \mathbb{Z}$, of the problem (1)-(3) are real and simple, and they can be numbered in ascending order on the real axis so that the corresponding angular function $\theta(x, \lambda_k)$ at $x = 0$ and $x = \pi$ satisfies the conditions*

$$\theta(0, \lambda_k) = \tau(\lambda_k) \text{ and } \theta(\pi, \lambda_k) = k\pi + \chi(\lambda_k),$$

respectively.

Using [2, Remark 5 and Theorem 2] from Theorem 1, we can find the number of zeros contained in the interval (0,1), the components of all eigenfunctions of the problem (1)-(3).

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The study of multidimensional mixed problem for one class of third order nonlinear equations

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This work is dedicated to the study of the existence in large of classical solutions of multidimensional mixed problems for one class of third-order semi-linear psevdohyperbolic equations. The conception of a classical solution for mixed problems under consideration is introduced. After applying the Fourier method, the solution of the original problem is reduced to the solution of some countable system of nonlinear inteqro-differential equations in unknown Fourier coefficients of the sought solution. Besides, the existence theorem in large of classical solution of the mixed problem is proved by the contracted mappings principle.

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Theses on four theorems regarding the surface of a right cylinder and a cone from place N. Tusi's Treatise "Comments on the work of Archimedes "On the ball and cylinder""

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In the "theorems" section of his comments on the work of Archimedes "On the ball and the cylinder", N. Tusi, in particular, under the numbers XIII (for Archimedes - XI) provides a proof of the following theorem: If there are two straight lines on the surface of a right "cylinder", then the surface of the cylinder between these lines there is more than a parallelogram, - enclosing between the lines located on the surface of the cylinder and the arcs connecting their ends.

Further, under the number XIV (in Archimedes-XII), Tusi comments on the proof of the following theorem: If there are two lines on the surface of any right cylinder and from the end of these lines to the circles that are the bases of the cylinder, some tangents are drawn that are in the plane of the bases and (pairwise) intersecting, then the parallelograms enclosed between the tangents and sides of the cylinder will be larger than the surface of the cylinder between both lines located on the surface of the cylinder.

The proof of the XIII theorem of Archimedes N. Tusi in his comments is given under the number XV: The surface of any right cylinder minus the bases is equal to a circle, the radius of which is the average proportional between the side of the cylinder and the diameter of its base.

Returning to the X theorem of Archimedes from the work "On the ball and the cylinder", which says that: If we draw tangents to the circle that is the base of the cone, which lie in the same plane with the circle and intersect each other, and then, the resulting points of tangency and the intersection of the tangents connect with the straight lines with the apex of the cone, then the triangles contained between the tangents and the lines connecting with the vertex of the cone will be larger than the cut-off or part of the conical surface. Tusi, in his comments, numbered this theorem under the number XII.

In the course of proving this theorem, N. Tusi proposed his own methodological amendment.

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Minimality conditions for Sturm-Liouville problems with a boundary condition depending affinely or quadratically on an eigenparameter

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In the paper we study Sturm-Liouville problems with a boundary condition depending affinely or quadratically on an eigenparameter:

$$-y'' + q(x)y = \lambda y,\ 0 < x < 1, \tag{1}$$

$$y'(0)\sin\beta = y(0)\cos\beta,\ 0 \le \beta < \pi, \tag{2}$$

$$y'(1) = (a\lambda^2 + b\lambda + c)y(1), \tag{3}$$

where λ is the spectral parameter, $q(x)$ is a real valued and continuous function on the interval $[0, 1]$, and a, b, c are real.

In the paper by Binding, Browne, Watson, and Code [9] the existence and asymptotics of eigenvalues of (1)-(3) were studied. It was proved that the eigenvalues of (1)-(3) form an infinite sequence, accumulating only at $+\infty$, and the following cases are possible: **(a)** All the eigenvalues are real and simple; **(b)** All the eigenvalues are simple and all, except a conjugate pair of non-real, are real; **(c)** All the eigenvalues are real and all, except one double, are simple; **(d)** All the eigenvalues are real and all, except one triple, are simple.

We define $y(x, \lambda)$ to be the non-zero solution of (1),(2), analytic in $\lambda \in \mathbf{C}$, and we write the characteristic equation as $\omega(\lambda) = y'(1, \lambda) - (a\lambda^2 + b\lambda + c)y(1, \lambda)$.

The eigenvalue λ_k is multiple if $\omega'(\lambda_k) = 0$, in particular, we say that λ_k is a double eigenvalue if in addition $\omega''(\lambda_k) \neq 0$, and a triple eigenvalue if $\omega''(\lambda_k) = 0 \neq \omega'''(\lambda_k)$.

Let y_n be an eigenfunction corresponding to eigenvalue λ_n. We denote by $(\cdot, \cdot)$ the scalar product in $L_2(0, 1)$, and by $\|\cdot\|_p$ the norm in $L_p(0, 1)$.

If λ_k is a multiple eigenvalue ($\lambda_k = \lambda_{k+1}$) then the first order associated function y_{k+1} is defined by $-y''_{k+1} + q(x)y_{k+1} = \lambda_k y_{k+1} + y_k$, $y'_{k+1}(0)\sin\beta = y_{k+1}(0)\cos\beta$, $y'_{k+1}(1) = (a\lambda_k^2 + b\lambda_k + c)y_{k+1}(1) + (2a\lambda_k + b)y_k(1)$.

If λ_k is a triple eigenvalue ($\lambda_k = \lambda_{k+1} = \lambda_{k+2}$) then together with the first order associated function y_{k+1} there exists the second order associated function y_{k+2}, for which the following relations hold: $-y''_{k+2} + q(x)y_{k+2} = \lambda_k y_{k+2} + y_{k+1}$, $y'_{k+2}(0)\sin\beta = y_{k+2}(0)\cos\beta$, $y'_{k+2}(1) = (a\lambda_k^2 + b\lambda_k + c)y_{k+2}(1) + (2a\lambda_k + b)y_{k+1}(1) + ay_k(1)$.

Definition 1. *If λ_k is a double eigenvalue then let $y^*_{k+1} = y_{k+1} + c_1 y_k$, where $c_1 = -\frac{\omega'''(\lambda_k)}{3\omega''(\lambda_k)} - \frac{\widehat{y}_{k+1}(1)}{y_k(1)}$, and $\widehat{y}_{k+1} = y_{k+1} - \widetilde{c}y_k$.*

Definition 2. *If λ_k is a triple eigenvalue then let $y^{\#}_{k+1} = y_{k+1} + c_2 y_k$ and $y^{\#}_{k+2} = y_{k+2} + c_2 y_{k+1} + d_1 y_k$, where $c_2 = -\frac{\omega^{IV}(\lambda_k)}{4\omega'''(\lambda_k)} - \frac{\widehat{y}_{k+1}(1)}{y_k(1)}$, $d_1 = -\frac{\omega^{V}(\lambda_k)}{20\omega'''(\lambda_k)} - \frac{\widehat{y}_{k+1}(1)\omega^{IV}(\lambda_k)}{4y_k(1)\omega'''(\lambda_k)} - \frac{\widehat{y}_{k+2}(1)\omega'''(\lambda_k)}{y_k(1)\omega'''(\lambda_k)} + c_2^2$, and $\widehat{y}_{k+2} = y_{k+2} - \widetilde{c}\widehat{y}_{k+1} - \widetilde{d}y_k$.*

The necessary and sufficient conditions for minimality and completeness of the chosen system of root functions of the corresponding operator were given in the form, which uses some special associated functions [3].

Theorem 1. *Let $a \neq 0$. **1)** In the case **(c)** the system $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k, j)$, where $j \neq k, k+1$ is arbitrary non-negative integer forms a minimal system in $L_2(0,1)$ if and only if $y^*_{k+1}(1)(\lambda_j - \lambda_k) \neq y_k(1)$.*

In the case **(d)** the system **2)** $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k, k+2)$ forms a minimal system in $L_2(0,1)$ if and only if $y^{\#}_{k+1}(1) \neq 0$. **3)** $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k, k+1)$ forms a minimal system in $L_2(0,1)$ if and only if $y^{\#}_{k+1}(1)^2 \neq y_k(1)y^{\#}_{k+2}(1)$. **4)** $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k+1, j)$, where $j \neq k, k+1, k+2$ is arbitrary non-negative integer forms a minimal system in $L_2(0,1)$ if and only if $y^{\#}_{k+1}(1)(\lambda_j - \lambda_k) \neq y_k(1)$. **5)** $\{y_n\}$ $(n = 0,\ 1, \ldots;\ n \neq k, j)$, where $j \neq k, k+1, k+2$ is arbitrary non-negative integer forms a minimal system in $L_2(0,1)$ if and only if $y^{\#}_{k+2}(1)(\lambda_j - \lambda_k) \neq y^{\#}_{k+1}(1)$.

In the current paper, another method with the direct use of characteristic functions was discussed. This direct method was known for the affine case [1,2] and was extensively discussed in the literature [4,6,7]. The aim of the present paper is to develop this direct method for the quadratic case and to consider the affine and quadratic cases together in a unified way.

Theorem 2. *In the case **(c)**, $y^*_{k+1}(1)(\lambda_j - \lambda_k) \neq y_k(1)$ if and only if $\widetilde{c} \neq \frac{1}{\lambda_j - \lambda_k} + \frac{1}{3} \cdot \frac{\omega'''(\lambda_k)}{\omega''(\lambda_k)}$.*

Theorem 3. *In the case **(d)**, $y^{\#}_{k+1}(1) \neq 0$ if and only if $\widetilde{c} \neq \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}$.*

Theorem 4. *In the case **(d)**, $y^{\#}_{k+1}(1)^2 \neq y_k(1)y^{\#}_{k+2}(1)$ if and only if*

$\widetilde{d} \neq \widetilde{c} \cdot \left(\widetilde{c} - \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right) + \frac{1}{20} \cdot \frac{\omega^{V}(\lambda_k)}{\omega'''(\lambda_k)}$.

Theorem 5. *In the case* ***(d)***, $y^{\#}_{k+1}(1)(\lambda_j - \lambda_k) \neq y_k(1)$ *if and only if* $\widetilde{c} \neq \frac{1}{\lambda_j - \lambda_k} + \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}$.

Theorem 6. *In the case* ***(d)***, $y^{\#}_{k+2}(1)(\lambda_j - \lambda_k) \neq y^{\#}_{k+1}(1)$ *if and only if* $\widetilde{d} \neq \left(\frac{1}{\lambda_j - \lambda_k} + \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right) \cdot \left(\widetilde{c} - \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right) + \frac{1}{20} \cdot \frac{\omega^{V}(\lambda_k)}{\omega'''(\lambda_k)}$.

Theorem 7. *In the case* ***(c)***, $y^{*}_{k+1}(1) \neq 0$ *if and only if* $\widetilde{c} \neq \frac{1}{3} \cdot \frac{\omega'''(\lambda_k)}{\omega''(\lambda_k)}$.

Theorem 8. *In the case* ***(d)***, $y^{\#}_{k+2}(1) \neq 0$ *if and only if* $\widetilde{d} \neq \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)} \cdot \left(\widetilde{c} - \frac{1}{4} \cdot \frac{\omega^{IV}(\lambda_k)}{\omega'''(\lambda_k)}\right) + \frac{1}{20} \cdot \frac{\omega^{V}(\lambda_k)}{\omega'''(\lambda_k)}$.

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On boyarsky-meyers inequality for a solution to the elliptic zaremba problem

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In the paper, we estimate solutions to the Zaremba problem for elliptic equations in bounded Lipschitz domain $D \in \mathbb{R}^n$, where $n > 1$, of the form

$$\mathcal{L}u := \operatorname{div}(a(x)\nabla u) \tag{1}$$

with uniformly elliptic measurable and symmetric matrix $a(x) = \{a_{ij}(x)\}$, i.e. $a_{ij} = a_{ji}$ and

$$\alpha^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \alpha|\xi|^2 \text{ for almost all } x \in D \text{ and for all } \xi \in \mathbb{R}^n. \tag{2}$$

Below we assume that the set $F \subset \partial D$ is closed and denote $G = \partial D \setminus F$. Denote by $W_2^1(D, F)$ the completion of the set of infinitely differentiable in the closure of D functions, vanishing in the vicinity of F, by the norm $\| u \|_{W_2^1(D,F)} = \left(\int\limits_D u^2\,dx + \int\limits_D |\nabla u|^2\,dx \right)^{1/2}$. Consider the Zaremba problem

$$\mathcal{L}u = l \text{ in } D, \quad u = 0 \text{ on } F, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } G, \tag{3}$$

where $\frac{\partial u}{\partial \nu}$ is an outer conormal derivative of u, and l is a linear functional in the space dual to $W_2^1(D, F)$. The functional l can be written as $l(\varphi) = -\sum\limits_{i=1}^{n} \int\limits_D f_i \varphi_{x_i}\,dx$, where $f_i \in L_2(D)$. The function $u \in W_2^1(D, F)$ is a solution to the problem (3) if the following integral identity holds

$$\int\limits_D a\nabla u \cdot \nabla\varphi\,dx = \int\limits_D f \cdot \nabla\varphi\,dx$$

for all functions $\varphi \in W_2^1(D, F)$.

The condition on the structure of the support of the Dirichlet data F plays a key role. For the formulation of the result, we need the notion of capacity. Let us define the capacity $C_p(K)$, $1 < p < n$, for the compact set $K \subset \mathbb{R}^n$ by the identity

$$C_p(K) = \inf \left\{ \int\limits_{\mathbb{R}^n} |\nabla \varphi|^p \, dx : \ \varphi \in C_0^\infty(\mathbb{R}^n), \ \varphi \geq 1 \text{ on } K \right\}.$$

Here and throughout $B_r^{x_0}$ is the open ball with radius r centered in the point x_0, and $mes_{n-1}(E)$ is $(n-1)$-measure of the set $E \subset \partial D$. Taking $p = 2n/(n+2)$ as $n > 2$ and $p = 3/2$ as $n = 2$ we assume one of the following conditions to be satisfied: for an arbitrary point $x_0 \in F$ as $r \leq r_0$ either the inequality

$$C_p(F \cap \overline{B}_r^{x_0}) \geq c_0 r^{n-p}, \tag{4}$$

or the inequality

$$mes_{n-1}(F \cap \overline{B}_r^{x_0}) \geq c_0 r^{n-1} \tag{5}$$

holds true, where the positive constant c_0 does not depend on x_0 and r. Condition (2) is stronger than (1), but it is more visual. Note that under one of these conditions the function $v \in W_2^1(D, F)$ satisfies the Friedrichs inequality

$$\int\limits_D v^2 \, dx \leq C \int\limits_D |\nabla v|^2 \, dx,$$

which leads to the unique solvability of problem (3) because of the Lax–Milgram theorem. To formulate the main result we give the definition of Lipschitz domain D in more detail.

A domain D will be called a Lipschitz domain, if for any point $x_0 \in \partial D$ there exists an open cube Q centered in x_0, faces of which are parallel to the coordinate axes, the length of the cube edges are independent of x_0, and in some cartesian coordinate system with origin in x_0 the set $Q \cap \partial D$ is a graph of the Lipschitz function $x_n = g(x_1, \ldots, x_{n-1})$ with the Lipschitz constant independent of x_0. Denote the length of the edge of such cubes by $2R_0$, and the Lipschitz constant of the respective functions g by L. For definiteness, we assume that the set $Q \cap D$ is located above the graph of the function g, and the constant r_0 from the conditions (4) and (5) is less than or equal to the constant R_0.

Theorem. *If $f \in L_{2+\delta_0}(D)$ with $\delta_0 > 0$, then there exist positive constants $\delta(n, \delta_0) < \delta_0$ and C such that for the solution to problem (3) the estimate*

$$\int\limits_D |\nabla u|^{2+\delta} dx \leq C \int\limits_D |f|^{2+\delta} \, dx \tag{6}$$

holds, where C depends only on δ_0, the dimension n, the ellipticity constant α from (2), the constant c_0 from (4) and (5), and also the constants L and R_0 from the definition of the Lipshcitz domain D.

Concerning Boyarsky-Meyers inequality, one can find the results, for instance, in [1], [2], [3], and [4].

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A dissipative singular boundary value transmission problem in limit-point case

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We consider the singular differential expression

$$\tau(y) := \frac{1}{\varrho(x)}\left[-\left(p(x)y'\right)' + q(x)y\right],\ x \in \Omega,$$

where $\Omega := \Omega_1 \cup \Omega_2$, $\Omega_1 := [a,c)$, $\Omega_2 := (c,b)$ and $-\infty < a < c < b \leq \infty$. We assume that the points a and c are regular and b is singular for the differential expression τ. The functions ϱ, p and q are real-valued, Lebesgue measurable functions on Ω and $\varrho, \frac{1}{p}, q \in L^1_{loc}(\Omega)$, $w(x) > 0$ for almost all $x \in \Omega$. The point c is regular if $\varrho, \frac{1}{p}, q \in L^1[c-\epsilon, c+\epsilon]$ for some $\epsilon > 0$. We assume that $\varrho(x), p(x)$ and $q(x)$ satisfy the Weyl's limit-point case conditions at singular endpoint b.

Let us consider the boundary value transmission problem

$$\tau\left(y\right) = \lambda y,\ x \in \Omega, \tag{1}$$

$$y^{[1]}\left(a\right) - \beta y\left(a\right) = 0, \tag{2}$$

$$y\left(c^{-}\right) = \alpha_1 y\left(c^{+}\right), \tag{3}$$

$$y^{[1]}\left(c^{-}\right) = \alpha_2 y^{[1]}\left(c^{+}\right), \tag{4}$$

where $y^{[1]}$ denotes py', α_1, α_2 are real numbers with $\alpha_1\alpha_2 > 0$ and β is some complex number such that $\beta = \Re\beta + i\Im\beta$ with $\Im\beta > 0$.

We consider the Hilbert space $\mathcal{H}$ with the inner product

$$\langle y, z\rangle_{\mathcal{H}} = \int_{\Omega_1} \varrho_1\left(x\right) y_1\left(x\right)\overline{z_1\left(x\right)}dx + \alpha_1\alpha_2 \int_{\Omega_2} \varrho_2\left(x\right) y_2\left(x\right)\overline{z_2\left(x\right)}dx,$$

where

$$y\left(x\right) = \begin{cases} y_1\left(x\right), & x \in \Omega_1, \\ y_2\left(x\right), & x \in \Omega_2, \end{cases} \quad y \in \mathcal{H},$$

$$z\left(x\right) = \begin{cases} z_1\left(x\right), & x \in \Omega_1, \\ z_2\left(x\right), & x \in \Omega_2, \end{cases} \quad z \in \mathcal{H},$$

and

$$\varrho(x)=\begin{cases}\varrho_1(x), & x\in\Omega_1,\\ \varrho_2(x), & x\in\Omega_2.\end{cases}$$

Let us consider the operator $\mathcal{S}_\beta$ with domain $\mathcal{D}(\mathcal{S}_\beta)$ consisting of vectors $y\in\mathcal{H}$ such that $y, y^{[1]}$ are locally absolutely continuous functions on Ω, $\tau(y)\in\mathcal{H}$, $R(y)=0$, $R_1(y)=0$, $R_2(y)=0$, where $R(y)=y^{[1]}(a)-\beta y(a)$, $R_1(y)=y(c^-)-\alpha_1 y(c^+)$ and $R_2(y)=y^{[1]}(c^-)-\alpha_2 y^{[1]}(c^+)$. Then we can handle the boundary value transmission problem $(1)-(4)$ in $\mathcal{H}$ as

$$\mathcal{S}_\beta y(x)=\lambda y(x),\ y\in\mathcal{D}(\mathcal{S}_\beta),\ x\in\Omega.$$

We shall remind that the linear operator T (with dense domain $\mathcal{D}(T)$) acting on some Hilbert Space H is called *dissipative* if $\Im(Tf,f)\geq 0$ for all $f\in\mathcal{D}(T)$ and *maximal dissipative* if it does not have any proper dissipative extension. Then we have:

Theorem 1. *The operator S_β is maximal dissipative in the Hilbert space $\mathcal{H}$.*

Recall that a linear operator $\mathbf{B}$ (with domain $\mathcal{D}(\mathbf{B})$) acting in a Hilbert space $\mathbf{H}$ is called *completely non-self-adjoint* (or *pure*) if the invariant subspace $\mathbf{M}\subseteq\mathcal{D}(\mathbf{B})$ $(\mathbf{M}\neq\{0\})$ of the operator $\mathbf{B}$ whose restriction to $\mathbf{M}$ is self-adjoint, does not exist.

Then we have the next conclusion.

Theorem 2. *The operator S_β is completely non-self-adjoint (pure).*

Let

$$\varphi(x,\lambda)=\begin{cases}\varphi_1(x,\lambda), & x\in\Omega_1,\\ \varphi_2(x,\lambda), & x\in\Omega_2,\end{cases}$$

$$\psi(x,\lambda)=\begin{cases}\psi_1(x,\lambda), & x\in\Omega_1,\\ \psi_2(x,\lambda), & x\in\Omega_2,\end{cases}$$

be the solutions of (1) satisfying the initial and transmission conditions

$$\begin{cases}\varphi_1(a,\lambda)=0, & \varphi_1^{[1]}(a,\lambda)=1,\\ \varphi_1(c^-,\lambda)=\alpha_1\varphi_2(c^+,\lambda), & \varphi_1^{[1]}(c^-,\lambda)=\alpha_2\varphi_2^{[1]}(c^+,\lambda),\end{cases}$$

and

$$\begin{cases}\psi_1(a,\lambda)=1, & \psi_1^{[1]}(a,\lambda)=0,\\ \psi_1(c^-,\lambda)=\alpha_1\psi_2(c^+,\lambda), & \psi_1^{[1]}(c^-,\lambda)=\alpha_2\psi_2^{[1]}(c^+,\lambda).\end{cases}$$

The *Weyl–Titchmarsh function* $M_\infty(\lambda)$ of the self-adjoint operator $\mathcal{S}_\infty$, generated boundary conditions $y(a) = 0$, $R_1(y) = 0$ and $R_2(y) = 0$, is uniquely determined from the condition

$$\psi(.,\lambda) + M_\infty(\lambda)\varphi(.,\lambda) \in \mathcal{H}, \Im\lambda \neq 0.$$

In this case $M_\infty(\lambda)$ is not in general a meromorphic function on $\mathbb{C}$, but is a holomorphic function with $\Im\lambda \neq 0$. The function $M_\infty(\lambda)$ has the key property $\Im\lambda\Im M_\infty(\lambda) > 0$ and $\overline{M_\infty(\lambda)} = M_\infty(\overline{\lambda})$, $\Im\lambda \neq 0$. Further, assume that the function $M_\infty(\lambda)$ is meromorphic in $\mathbb{C}$. This condition is equivalent to the fact that the operator $\mathcal{S}_\infty$ has a purely discrete spectrum. Then we have

Theorem 3. *The spectrum of the boundary value transmission problem (1)-(4) (or operator $\mathcal{S}_\beta$) is purely discrete and belongs to the open upper half-plane. For all values of βith $\Im\beta > 0$,except possibly for a single value $\beta = \beta_0$,the boundary value transmission problem (1)-(4) (or operator $\mathcal{S}_\beta$)has a countable number of isolated eigenvalues with finite algebraic multiplicity and limit points at infinity. The system of all eigenfunctions and associated functions (or all root functions) of this problem ($\beta \neq \beta_0$) is complete in the space $\mathcal{H}$.*

Reconstruction of the Sturm-Liouville operator from nodal data

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One of the solution methods for the inverse problems of the Sturm-Liouville operators is to use the zeros of the eigenfunctions. These zeros are also called nodal points. Trying to reconstruct the coefficients of the operator from the asymptotic formula of the nodal points is known as an inverse nodal problem. This problem for Sturm-Liouville operator was first investigated by McLaughlin in [1]. She accomplished to prove that this type of inverse problem has a unique solution. In 1997, F.-C. Yang [2] obtained a definite algorithm for the solution of inverse nodal problems with separated boundary conditions. Next, inverse nodal problems for Sturm-Liouville operators with discontinuous conditions were first intestigated by Chung-Tsun Shieh and V.A. Yurko in [3]. In this study, the uniqueness theorem is proved for the solution of the inverse nodal problem to determine the potential function when the discontinuity point is the midpoint of the segment. However, the solution of the nodal inverse problem is not given when the point of discontinuity is any point in the interval.

In this paper, unlike previous studies on this subject, when the discontinuity point $(0,\pi)$ is any of the countable number of irrational points in the form of $d_r = r\pi,\ (r \in (0,1) \cap \mathbb{Q})$, the proof of the uniqueness theorem is given for the solution of the inverse nodal problem and give an algorithm for the reconstruction of the coefficients of the problem using asymptotics of the nodal points.

Consider the following boundary value problem $L = L\left(q, h, H, a_1, a_2, d\right)$ with discontinuity conditions inside the interval:

$$-y^{''} + q(x)y = \lambda y, \qquad 0 < x < \pi, \tag{1}$$

$$U(y) := y^{'}\left(0\right) - hy\left(0\right) = 0, V(y) := y^{'}\left(\pi\right) - Hy\left(0\right) = 0, \tag{2}$$

$$y\left(d+0\right) = a_1 y\left(d-0\right), \quad y^{'}\left(d+0\right) = a_1^{-1} y^{'}\left(d-0\right) + a_2\, y(d-0). \tag{3}$$

Here $d \in \Re := \{r\pi, r \in (0,1) \cap \mathbb{Q}\}$, λ is the spectral parameter, $q(x)$ is a real-valued function, h, H, a_1, a_2 are real numbers, $q(x) \in L(0,\pi)$ and $a_1 > 0$.

Without loss of generality, we assume that

$$\int_0^\pi q(x)dx = 0. \tag{4}$$

Let $\varphi(x, \lambda_n)$ be the eigenfunctions of the problem L. For the boundary value problem L an analog of Sturm's oscillation theorem is true. More precisely, the eigenfunction $\varphi(x, \lambda_n)$ has exactly n (simple) zeros inside the interval $(0, \pi)$, namely: $0 < x_n^1 < x_n^2 < ... < x_n^n < T$.

The set $X(L) := \left\{x_n^j : n = 1, 2, ..., j = \overline{1, n}\right\}$ is called the set of nodal points of the boundary value problem L.

Let $X_0(L) := \left\{x_{n_k}^{j(n_k)} : n_k = 1, 2, ..., j(n_k) = \overline{1, n_k}\right\}$ be a subsequence of the numbers $x_n^{j(n)}$ that is dense on $(0, \pi)$.

Consider the problem $\widetilde{L} = L\left(\widetilde{q}, \widetilde{h}, H, a_1, a_2, d\right)$ under the same assumption with L.

Theorem 1. If $X_0(L) = X_0\left(\widetilde{L}\right)$, $H = \widetilde{H}$, then for any $d \in \mathbb{R}$, $q(x) = \widetilde{q}(x)$ a.e. on $(0, \pi)$, and $h = \widetilde{h}$.

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A study on parametric soft S-metric spaces

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Throughout this paper, we mention the concept of parametric soft $S-$metric space and give some fixed soft point results for soft contraction mapping in complete parametric soft $S-$metric space.

Definition 1. *A parametric soft $S-$metric on $\widetilde{X}$ is a mapping $d_S : SP(\widetilde{X}) \times SP(\widetilde{X}) \times SP(\widetilde{X}) \times \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ that satisfies the following conditions*

S1) $d_S\left(x_a, y_b, z_c, \widetilde{t}\right) \widetilde{\geq} \widetilde{0}$,

S2) $d_S\left(x_a, y_b, z_c, \widetilde{t}\right) = \widetilde{0}$ *if and only if* $x_a = y_b = z_c$,

S3) $d_S\left(x_a, y_b, z_c, \widetilde{t}\right) \widetilde{\leq} d_S\left(x_a, x_a, u_d, \widetilde{t}\right) + d_S\left(y_b, y_b, u_d, \widetilde{t}\right) + d_S\left(z_c, z_c, u_d, \widetilde{t}\right)$

for each soft points $x_a, y_b, z_c, u_d \in SP(\widetilde{X})$ *and* $\widetilde{t} > \widetilde{0}$.

Then the soft set $\widetilde{X}$ with a parametric soft $S-$metric d_S is called a parametric soft $S-$metric space and denoted by $\left(\widetilde{X}, d_S, E\right)$.

Definition 2. *Let $\left(\widetilde{X}, d_S, E\right)$ be a parametric soft $S-$metric space. $f_\varphi : \left(\widetilde{X}, d_S, E\right) \to \left(\widetilde{X}, d_S, E\right)$ is said to be a soft contraction mapping in parametric soft $S-$metric space, if there exists a soft real number $\widetilde{q} \in \mathbb{R}(E), \widetilde{0} \widetilde{\leq} \widetilde{q} \widetilde{<} \widetilde{1}$ such that*

$$d_S\left(f_\varphi\left(x_a\right), f_\varphi\left(x_a\right), f_\varphi\left(y_b\right), \widetilde{t}\right) \widetilde{\leq} \widetilde{q} d_S\left(x_a, x_a, y_b, \widetilde{t}\right)$$

for all $x_a, y_b \in SP(\widetilde{X})$ *and* $\widetilde{t} > \widetilde{0}$.

Lemma. *Let $\left(\widetilde{X}, d_S, E\right)$ be a parametric soft $S-$metric space. Then*

$$d_S\left(x_a, x_a, y_b, \widetilde{t}\right) = d_S\left(y_b, y_b, x_a, \widetilde{t}\right)$$

for all $x_a, y_b \in SP(\widetilde{X})$ *and* $\widetilde{t} > \widetilde{0}$.

Theorem. *Let* $\left(\widetilde{X}, d_S, E\right)$ *be a complete parametric soft* $S-$*metric space and* $f_\varphi : \left(\widetilde{X}, d_S, E\right) \to \left(\widetilde{X}, d_S, E\right)$ *be a surjective soft self-mapping. If there is soft real numbers* $\widetilde{r}, \widetilde{s} \geq \widetilde{0}$ *and* $\widetilde{k} > \widetilde{1}$ *such that*

$$d_S\left(f_\varphi(x_a), f_\varphi(x_a), f_\varphi(y_b), \widetilde{t}\right) \geq$$
$$\geq \widetilde{k} d_S\left(x_a, x_a, y_b, \widetilde{t}\right) + \widetilde{r} d_S\left(f_\varphi(x_a), f_\varphi(x_a), x_a, \widetilde{t}\right) +$$
$$+\widetilde{s} d_S\left(f_\varphi(y_b), f_\varphi(y_b), y_b, \widetilde{t}\right)$$

for each $x_a, y_b \in SP(\widetilde{X})$ *and* $\widetilde{t} > \widetilde{0}$, *then* f_φ *has a unique fixed soft point in* $SP(\widetilde{X})$.

Corollary 1. *Let* $\left(\widetilde{X}, d_S, E\right)$ *be a complete parametric soft* $S-$*metric space and* $f_\varphi : \left(\widetilde{X}, d_S, E\right) \to \left(\widetilde{X}, d_S, E\right)$ *be a surjective soft self-mapping. If there is soft real numbers* $\widetilde{r} \geq \widetilde{0}$ *and* $\widetilde{k} > \widetilde{1}$ *such that*

$$d_S\left(f_\varphi(x_a), f_\varphi(x_a), f_\varphi(y_b), \widetilde{t}\right) \geq \widetilde{k} d_S\left(x_a, x_a, y_b, \widetilde{t}\right)$$
$$+\widetilde{r} \max\left\{d_S\left(f_\varphi(x_a), f_\varphi(x_a), x_a, \widetilde{t}\right), d_S\left(f_\varphi(y_b), f_\varphi(y_b), y_b, \widetilde{t}\right)\right\}$$

for each $x_a, y_b \in SP(\widetilde{X})$ *and* $\widetilde{t} > \widetilde{0}$, *then* f_φ *has a unique fixed soft point in* $SP(\widetilde{X})$.

Corollary 2. *Let* $\left(\widetilde{X}, d_S, E\right)$ *be a complete parametric soft* $S-$*metric space and* $f_\varphi : \left(\widetilde{X}, d_S, E\right) \to \left(\widetilde{X}, d_S, E\right)$ *be a surjective soft self-mapping. If there is soft real number* $\widetilde{m} > \widetilde{1}$ *such that*

$$d_S\left(f_\varphi(x_a), f_\varphi(x_a), f_\varphi(y_b), \widetilde{t}\right) \geq \widetilde{m} d_S\left(x_a, x_a, y_b, \widetilde{t}\right)$$

for each $x_a, y_b \in SP(\widetilde{X})$ *and* $\widetilde{t} > \widetilde{0}$, *then* f_φ *has a unique fixed soft point in* $SP(\widetilde{X})$.

Corollary 3. *Let* $\left(\widetilde{X}, d_S, E\right)$ *be a complete parametric soft* $S-$*metric space and* $f_\varphi : \left(\widetilde{X}, d_S, E\right) \to \left(\widetilde{X}, d_S, E\right)$ *be a surjective soft self-mapping. If there is a positive integer* n *and a soft real number* $\widetilde{k} > \widetilde{1}$ *such that*

$$d_S\left(f_\varphi^n(x_a), f_\varphi^n(x_a), f_\varphi^n(y_b), \widetilde{t}\right) \geq \widetilde{k} d_S\left(x_a, x_a, y_b, \widetilde{t}\right)$$

for each $x_a, y_b \in SP(\widetilde{X})$ *and* $\widetilde{t} > \widetilde{0}$, *then* f_φ *has a unique fixed soft point in* $SP(\widetilde{X})$.

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Some fixed- point type theorems on parametric soft $b-$ metric spaces

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We firstly introduce the concepts of parametric soft $b-$metric space and parametric soft $S-$metric space. Also, we investigate some relationships between parametric soft metric, parametric soft $S-$metric and parametric soft $b-$metric. Later we give the existence and uniqueness of some fixed soft points of continuous and surjective mapping satisfying contractive condition in complete parametric soft $b-$metric space. Let $\widetilde{X}$ be the absolute soft set, E be a non-empty set of parameters and $SP(\widetilde{X})$ be the collection of all soft points of $\widetilde{X}$. Let $\mathbb{R}(E)^*$ denote the set of all non-negative soft real numbers.

Definition 1. A parametric soft $S-$metric on $SP(\widetilde{X})$ is a mapping $d_S : SP(\widetilde{X}) \times SP(\widetilde{X}) \times SP(\widetilde{X}) \times \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ that satisfies the following conditions:

S1) $d_S\left(x_a, y_b, z_c, \widetilde{t}\right) \widetilde{\geq} \widetilde{0}$,

S2) $d_S\left(x_a, y_b, z_c, \widetilde{t}\right) = \widetilde{0}$ if and only if $x_a = y_b = z_c$,

S3) $d_S\left(x_a, y_b, z_c, \widetilde{t}\right) \widetilde{\leq} d_S\left(x_a, x_a, u_d, \widetilde{t}\right) + d_S\left(y_b, y_b, u_d, \widetilde{t}\right) + d_S\left(z_c, z_c, u_d, \widetilde{t}\right)$

for each soft points $x_a, y_b, z_c, u_d \in SP(\widetilde{X})$ and all $\widetilde{t} > \widetilde{0}$.

Then the soft set $\widetilde{X}$ with a parametric soft $S-$metric d_S is called a parametric soft $S-$metric space and denoted by $\left(\widetilde{X}, d_S, E\right)$.

Lemma 1. Let $\left(\widetilde{X}, d_S, E\right)$ be a parametric soft $S-$metric space. Then

$$d_S\left(x_a, x_a, y_b, \widetilde{t}\right) = d_S\left(y_b, y_b, x_a, \widetilde{t}\right)$$

for all $x_a, y_b \in SP(\widetilde{X})$ and $\widetilde{t} > \widetilde{0}$.

Definition 2. A parametric soft $b-$metric on $SP(\widetilde{X})$ is a mapping $b_S : SP(\widetilde{X}) \times SP(\widetilde{X}) \times \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ that satisfies the following conditions:

b1) $b_S\left(x_a, y_b, \widetilde{t}\right) = \widetilde{0}$ if and only if $x_a = y_b$,

b2) $b_S\left(x_a, y_b, \widetilde{t}\right) = b_S\left(y_b, x_a, \widetilde{t}\right)$,
b3) $b_S\left(x_a, y_b, \widetilde{t}\right) \widetilde{\leq} \widetilde{s}\left[b_S\left(x_a, z_c, \widetilde{t}\right) + b_S\left(z_c, y_b, \widetilde{t}\right)\right]$
for each soft points $x_a, y_b, z_c \in SP(\widetilde{X})$ and all $\widetilde{t} > \widetilde{0}, \widetilde{s} \geq \widetilde{1}$.

Then the soft set $\widetilde{X}$ with a parametric soft $b-$metric b_S is called a parametric soft $b-$metric space and denoted by $\left(\widetilde{X}, b_S, \widetilde{s}, E\right)$.

Definition 3. Let $\left(\widetilde{X}, b_S, \widetilde{s}, E\right)$ be a parametric soft $b-$metric space and $\left\{x_{a_n}^n\right\}$ be a soft sequence of soft points in $\widetilde{X}$.

(i) The soft sequence $\left\{x_{a_n}^n\right\}$ is called convergent to x_a,written as $\lim\limits_{n\to\infty} x_{a_n}^n = x_a$, if $\lim\limits_{n\to\infty} b_S\left(x_{a_n}^n, x_a, \widetilde{t}\right) = \widetilde{0}$ for all $\widetilde{t}\widetilde{>}\widetilde{0}$.

(ii) The soft sequence $\left\{x_{a_n}^n\right\}$ is called a Cauchy sequence if $\lim\limits_{n,m\to\infty} d_S\left(x_{a_n}^n, x_{a_m}^m, \widetilde{t}\right) = \widetilde{0}$ for all $\widetilde{t}\widetilde{>}\widetilde{0}$.

(iii) A parametric soft $b-$metric $\left(\widetilde{X}, b_S, \widetilde{s}, E\right)$ is called complete if every Cauchy sequence is convergent.

Definition 4. Let $\left(\widetilde{X}, b_S, \widetilde{s}, E\right)$ be a parametric soft $b-$metric space and $f_\varphi : \left(\widetilde{X}, b_S, \widetilde{s}, E\right) \to \left(\widetilde{X}, b_S, \widetilde{s}, E\right)$ be a soft mapping. Then f_φ is a soft continuous mapping at soft point x_a in $\widetilde{X}$, if for any soft sequence $\left\{x_{a_n}^n\right\}$ in $\widetilde{X}$ such that $\lim\limits_{n\to\infty} b_S\left(x_{a_n}^n, x_a, \widetilde{t}\right) = \widetilde{0}$, then $\lim\limits_{n\to\infty} b_S\left(f_\varphi\left(x_{a_n}^n\right), f_\varphi\left(x_a\right), \widetilde{t}\right) = \widetilde{0}$ is satisfied.

Theorem 1. Let $\left(\widetilde{X}, b_S, \widetilde{s}, E\right)$ be a complete parametric soft $b-$metric space and $f_\varphi : \left(\widetilde{X}, b_S, \widetilde{s}, E\right) \to \left(\widetilde{X}, b_S, \widetilde{s}, E\right)$ be a continuous mapping satisfying the following condition

$$\begin{aligned} & b_S\left(f_\varphi\left(x_a\right), f_\varphi\left(y_b\right), \widetilde{t}\right) + \widetilde{\alpha} \max\left\{b_S\left(x_a, f_\varphi\left(y_b\right), \widetilde{t}\right), b_S\left(y_b, f_\varphi\left(x_a\right), \widetilde{t}\right)\right\} \\ \geq \ & \widetilde{\beta} \frac{b_S\left(x_a, f_\varphi\left(x_a\right), \widetilde{t}\right)\left[\widetilde{1} + b_s\left(y_b, f_\varphi\left(y_b\right), \widetilde{t}\right)\right]}{\widetilde{1} + b_s\left(x_a, y_b, \widetilde{t}\right)} + \widetilde{\gamma} b_s\left(x_a, y_b, \widetilde{t}\right) \qquad (1) \end{aligned}$$

for all $x_a, y_b \in SP(\widetilde{X})$, $x_a \neq y_b$ and for all $\widetilde{t} > \widetilde{0}$, where $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma} \geq \widetilde{0}$ are soft real constants and $\widetilde{s}\widetilde{\beta} + \widetilde{\gamma} > \left(\widetilde{1} + \widetilde{\alpha}\right)\widetilde{s} + \widetilde{s}^2\widetilde{\alpha}$, $\widetilde{\gamma} > \left(\widetilde{1} + \widetilde{\alpha}\right)$.Then f_φ has a unique fixed soft point in $SP(\widetilde{X})$.

Theorem 2. Let $\left(\widetilde{X}, b_S, \widetilde{s}, E\right)$ be a complete parametric soft $b-$metric space and $f_\varphi : \left(\widetilde{X}, b_S, \widetilde{s}, E\right) \to \left(\widetilde{X}, b_S, \widetilde{s}, E\right)$ be a surjective mapping satisfying the condition (1) $x_a, y_b \in SP(\widetilde{X})$ and for all $\widetilde{t} > \widetilde{0}$, where $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma} \geq \widetilde{0}$ are soft real constants and $\widetilde{s}\widetilde{\beta} + \widetilde{\gamma} > \left(\widetilde{1} + \widetilde{\alpha}\right)\widetilde{s} + \widetilde{s}^2\widetilde{\alpha}$, $\widetilde{\gamma} > \left(\widetilde{1} + \widetilde{\alpha}\right)$.Then f_φ has a unique fixed soft point in $SP(\widetilde{X})$.

Theorem 3. Let $f_\varphi, g_\psi : \left(\widetilde{X}, b_S, \widetilde{s}, E\right) \to \left(\widetilde{X}, b_S, \widetilde{s}, E\right)$ be two surjective mappings of a complete parametric soft $b-$metric space $\left(\widetilde{X}, b_S, \widetilde{s}, E\right)$. Consider that f_φ, g_ψ satisfying the following conditions:

$$b_S\left(f_\varphi\left(g_\psi\left(x_a\right)\right), g_\psi\left(x_a\right), \widetilde{t}\right) + \widetilde{k}b_S\left(f_\varphi\left(g_\psi\left(x_a\right)\right), x_a, \widetilde{t}\right) \geq \widetilde{\alpha}b_S\left(g_\psi\left(x_a\right), x_a, \widetilde{t}\right)$$

and

$$b_S\left(g_\psi\left(f_\varphi\left(x_a\right)\right), f_\varphi\left(x_a\right), \widetilde{t}\right) + \widetilde{k}b_S\left(g_\psi\left(f_\varphi\left(x_a\right)\right), x_a, \widetilde{t}\right) \geq \widetilde{\beta}b_S\left(f_\varphi\left(x_a\right), x_a, \widetilde{t}\right)$$

for all $x_a \in SP(\widetilde{X})$, for all $\widetilde{t} > \widetilde{0}$ and $\widetilde{\alpha}, \widetilde{\beta}, \widetilde{k} \in \mathbb{R}(E)^*$ with $\widetilde{\alpha} > \widetilde{s}\left(\widetilde{1} + \widetilde{k}\right) + \widetilde{s}^2\widetilde{k}$ and $\widetilde{\beta} > \widetilde{s}\left(\widetilde{1} + \widetilde{k}\right) + \widetilde{s}^2\widetilde{k}$. If f_φ or g_ψ is soft continuous mapping, then f_φ and g_ψ also have a common fixed soft point.

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Mathematical modeling of processes in one and two componential nonlinear maedia

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For mathematical modeling of nonlinear processes described by nonlinear degenerate type parabolic equation and system for computational goal very important before numerical analysis of a solution of nonlinear problem as showed practice necessary to investigate the qualitative properties of solutions such us global solvability, non solvability, a condition of arising blow up solution, estimate, asymptotic of solution, to find an exact solution (if it possible) depending on value of numerical parameters of considered problems. Below we demonstrate this approach for the investigation problem Cauchy on example modeling of nonlinear processes described one and two componential media.

Consider the following Cauchy problem in the domain $Q = \{(t,x) : t > 0,\ x \in R^N\}$ for the double nonlinear time dependent degenerate parabolic system

$$\begin{aligned} L_1(u,v) &= -|x|^{-l}\tfrac{\partial u}{\partial t} + div\left(|x|^n v^{m_1-1}\left|\nabla u^k\right|^{p-2}\nabla u\right) + \varepsilon|x|^{-l}\gamma(t)v^{\beta_1} = 0 \\ L_2(u,v) &= -|x|^{-l}\tfrac{\partial v}{\partial t} + div\left(|x|^n u^{m_2-1}\left|\nabla v^k\right|^{p-2}\nabla v\right) + \varepsilon|x|^{-l}\gamma(t)u^{\beta_2} = 0,\ \varepsilon = \pm 1 \end{aligned} \quad (1)$$

$$u(0,x) = u_0(x) \ge 0, \qquad v(0,x) = v_0(x) \ge 0, \qquad x \in R^N \quad (2)$$

where $k \ge 1$, $p \ge 2$, $m_i \ge 1$, $\beta_i > 0 \quad i = 1,2$, are the given numerical parameters,$\nabla(.) = grad(.)$,$0 \le u_0(x),\ v_0(x) \in C(R^N)$ are given functions, $0 < \gamma(t) \in C(0,\infty)$ The equation (1) is a base for modeling of the many physical processes [1-4], for example this system describe the processes of reaction - diffusion, heat conductivity, polytrophic filtration of gas and liquid ($p = 2$) in the nonlinear media with source power of which is equal to $\gamma(t)u^{\beta_i},\ i = 1,2$. A specifically property of this equation is it degenerating. In the domain where $u = 0$ or $\nabla u = 0$ system (1) are degenerated to an equation of the first order. Therefore, we need to investigate the weak solution, because in this case solutions of (1) may do not exist in the classical sense. One of universally method for establishing the new nonlinear effects for considering problem as shown in many works is self-similar, approximately self-similar approach [1-4]. This approach intensively used for investigation such properties of solution as a finite speed of perturbation, a blow up, a space localization of solutions and so on. For construction of the self-similar, an approximately self-similar solution we use method of nonlinear splitting [4]

The system (1) in the domain where $u\,(t,\,x)\equiv 0$ and $v\,(t,\,x)\equiv 0$ are degenerate type parabolic system.

Therefore for the solution of the system (1) have place phenomena of the finite speed of a propagation (FSP) i.e. there are the functions $l_1\,(t)\,,\;\; l_2\,(t)$, that $u\,(t,\,x)\equiv 0$ and $v\,(t,\,x)\equiv 0$ when $|x|\geq l_1\,(t)$ and $|x|\geq l_2\,(t)$. In the case $l_1\,(t)\,,l_2\,(t)<\infty$ for $\forall t>0$ solution of the problem (1),(2) is called a space localization of a perturbation (SLP). The surfaces $|x|=l_1\,(t)$ and $|x|=l_2\,(t)$ are called a free boundary or a front.

In the point of physics view it is reasonable to consider weak solution that has properties of the bounded, continuity, and satisfying to the conditions

$$0\leq u(t,x),\;\; v(t,x), |x|^n u^{m_1-1}|\nabla u|^{p-2}\nabla u,\;\; |x|^n v^{m_2-1}|\nabla v|^{p-2}\nabla v\in C(Q),$$

and satisfy to some integral identity.

In the talk based on the self similar and approximately self similar solutions of the system (1) the qualitative properties of solutions of the considered system (1) under action of source and variable density discussed. It is shown that properties of the solutions problem (1) depends on value of the numerical parameters characterizing nonlinear media, value of variable density and source or absorption are different. For the case $m_i+k(p-2)-1>0,\;\; i=1,2$ based on approximately self-similar system equation constructed using the algorithm of a nonlinear splitting [4] the Fujita type global solvability and an estimate of the front [3], an asymptotic behavior of the compactly supported solutions of the considered problem and a behavior of a free boundary for the case is established. It is proved an existing of a solution with property of a finite speed of perturbations, localization of solution, a blow up. An asymptotic of a self-similar solution for the fast diffusion case $m_i+k(p-2)-1<0,\;\; i=1,2$ and a critical $m_i+k(p-2)-1=0,\;\; i=1,2$ cases is studied. It is showed that in singular case when $p=n+l$ the solution of the problem (1), (2) have logarifmical singularity near of the point $x=0$. It is shown that the coefficients of principal member of the asymptotic of the solution of the self-similar system satisfied to some system of a nonlinear algebraic equation. Based on these qualitative properties of the solution, using the approximately self similar solutions for the initial approximation the numerical experiments, visualization of processes described by the system reaction diffusion carried out.

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Property of external curvature of convex surfaces with a finite number of vertices

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In [1], A. D. Alexander defined the concept of a limit cone of a convex surface and, depending on the type of limit cone, classified points on a convex surface. A surface point is regular if the limiting cone degenerates to a plane, edged if the limiting cone degenerates into a dihedral angle, conical when the limiting cone is not degenerate.

If there are no ribbed and conical points on the surface, it is called regular. In polyhedral, regular points are its faces, edged points and vertices coincide with the edges and vertices of the polyhedron.

Definition. If a surface does not have edged points, but has a finite number of vertices, it is called a surface with a finite number of vertices.

The concept of a spherical mapping of a polyhedron was introduced and its properties were studied in [1]. The external curvature of a set on a convex polyhedron is also defined as the area of its spherical mapping.

We have revealed some properties of the external curvature of a convex surface with a finite number of vertices.

Let F be a surface projecting uniquely onto a convex surface G with the boundary ∂G. The boundary of the surface γ projects uniquely onto the boundary of the surface ∂G. Points $A_1, A_2, ..., A_k \in G$ and these points separate from the edge ∂G. We denote by $W(G)$ the class of convex surfaces F with vertices projecting to points A_i, and regular at other points of the surfaces G with the boundary γ.

Consider a set $M \subset F \in W(G)$ with projection $M' \subset G$, and spherical mapping M^* on the unit sphere. We denote by $\omega_F(M)$ the external curvature, that is, the area $S(M^*)$ of the spherical mapping.

Since the surface F has a vertex at the point $A_i' \in F$ projecting into the points A_i, these points are conical.

We denote by ω_i the value of the external curvature of the surface at the point $X_i' \in F$.

Theorem. *The external curvature of the set $\omega_F(M)$ has the following properties:*

1. The external curvature of a set $M \subset F$ is a positive definite, additive function of a set.

2. If the set M contains k vertices X_i, then $\omega_F(M) \geq \sum_{i=1}^{k} \omega_i$.

3. If the sequence of sets $M_n \subset W$ containing the vertex X_m tends to the point X_m, then $\lim\limits_{M_n \to X_m} \omega_F(M_n) = \omega_M$.

The proof of these properties is based on the properties of the spherical mapping of a convex surface.

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One class of non-classical nonlinear Volterra integral equation of the first kind

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Consider a nonclassical nonlinear integral equation of the first kind

$$\int_{\alpha(t)}^{t} K(t,s,u(s))ds = f(t), t \in [t_0, T], \tag{1}$$

where $\alpha(t) \in C[t_0,T], \alpha(t_0) = t_0, \alpha(t) \leq t, t \in [t_0,T], f(t)$ and $K(t,s,u(s))$ -are known functions in the domains $[t_0,T]$ and $G \times R$, respectively, $G = \{(t,s) : t_0 \leq t \leq T, \alpha(t) \leq s \leq t\}$, $f(t_0) = 0$ and $u(t)-$ is the required function.

The theory and applications of integral equations were studied in numerous works. Thus, in particular, a survey of the results of investigations of the Volterra integral equations of the second kind is presented in [1]. The Volterra integral equations of the first and third kinds with smooth kernels were studied in [2]. In [3], Lavrentaev's regularizing operators were constructed for the solution of linear Fredholm integral equations of the first kind. The theory and numerical methods used for the solution of nonclassical linear Volterra integral equations of the first kind with differentiable and nonzero kernels on a diagonal were analyzed in [4]. In [4-7], one can find the applications of nonclassical Volterra integral equations of the first kind to various practical problems. In [8], by using Lavrentaev's regularization method, approximate solutions were constructed for the Volterra integral equations of the first kind with smooth and nonzero kernels on the diagonal and differentiable solutions. The theorems on the uniqueness of solutions were proved and regularizing operators for the solutions of systems of linear and nonlinear Volterra integral equations of the first and third kinds were constructed in [9, 10]. In [11], the uniqueness theorem was proved for a system of linear integral Fredholm equations of the third kind and a regularizing operator was constructed for the solution of this system. On the basis of a new approach, the problems of existence and uniqueness of the solutions of Fredholm integral equations of the third kind with multipoint singularities and systems of these equations were investigated in [12, 13]. A survey of the results of investigations of Volterra integral equations of the first kind can be found in [14].In [15], on the basis of a modification of the research method proposed in [9], regularizing Lavrentiev operators were constructed and uniqueness theorems were proved for solving nonclassical linear integral Volterra equations of the first kind with undifferentiated kernels, and the kernels on the diagonal can be zero in finite numbers of points.

In the present paper, on the basis of a modified method proposed in [9], we prove unique theorems and construct Lavrentev's regularizing operators for the solutions of nonclassical nonlinear Volterra integral equations of the first kind (1).

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Representation by a sum of algebras in the space of continuous functions

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Assume X is a compact Hausdorff space and $C(X)$ is the space of continuous real-valued functions on X. In 1948, M. Stone [6] showed that a closed subalgebra $A \subset C(X)$, which contains a nonzero constant function, coincides with the whole space $C(X)$ if and only if A separates points of X.

Assume we are given closed subalgebras $A_1, ..., A_k$ of $C(X)$. Obviously, the condition of separation of points remains necessary also for the equality $A_1 + \cdots + A_k = C(X)$, $k > 2$. But this condition and its subsequent known refinements such as *strong separation of points*, *uniform separation of points* are far from being sufficient. A sufficient and at the same time geometrically explicit condition for the above representation problem in a compact metric space X was first obtained by Sternfeld [5] in 1978. In 1992, Sproston and Straus [4] proved that Sternfeld's result is valid for a compact Hausdorff space. The purpose of our talk is to give a necessary condition of such type for the representation $C(X) = A_1 + \cdots + A_k$. We see that in the particular case of two subalgebras ($k = 2$), our necessary condition turns out to be also sufficient. As an application, we see that the classical Stone-Weierstrass theorem easily follows from our result.

The above problem of representation is closely related to the Kolmogorov superposition theorem [2]. A generalization of this theorem, due to Ostrand [3], states that for the $d-$dimensional compact metric space Y, the space $C(Y)$ can be represented as a sum of its $2d + 1$ closed subalgebras, each generated by one fixed element $h_i \in C(Y)$, $i = 1, ..., 2d + 1$. For a detailed information about Kolmogorov's superposition theorem and its extensions see Ismailov [1].

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Non-local and inverse problems for the Rayleigh-Stokes equation

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Our report is based on joint work [1], [2] and [3] with our colleagues: S.R. Umarov (New Haven University, USA), A. Mukhiddinova (Tashkent University of Information Technologies), and N. Vaisova (Institute of Mathematics, Uzbekistan Academy of Science, Urgancg).

In this talk we consider the following Rayleigh-Stokes problem for a generalized second-grade fluid with a time-fractional derivative model:

$$\partial_t u(x,t) - (1+\gamma\,\partial_t^\alpha)\Delta u(x,t) = f(x,t), \quad x\in\Omega, \quad 0<t\le T;$$

$$u(x,t)=0, \quad x\in\partial\Omega, \quad 0<t\le T;$$

$$u(x,0)=\varphi(x), \quad x\in\Omega,$$

where $1/\gamma>0$ is the fluid density, a fixed constant, φ and f are given functions, $\partial_t=\partial/\partial t$, and ∂_t^α is the Riemann-Liouville fractional derivative of the order $\alpha\in(0,1)$ defined by:

$$\partial_t^\alpha h(t) = \frac{d}{dt}\int_0^t \omega_{1-\alpha}(t-s)h(s)ds, \quad \omega_\alpha(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}. \tag{1}$$

Here $\Gamma(\sigma)$ is Euler's gamma function. Based on physical considerations, usually, this problem is considered in the domain $\Omega\subset R^N$, $N=1,2,3$, with a sufficiently smooth boundary $\partial\Omega$.

The main results of this talk are the following:

1) We will solve the Rayleigh-Stokes problem by the Fourier method. A formal formula for the solution in the form of a Fourier series was given in the papers of previous authors, but the convergence of these series was not investigated.

2) We will pay special attention to the backward since in previous papers the authors considered only the case $N\le 3$. And this is connected with the method used in these works: if the dimension of the space is less than four, then

for the eigenvalues λ_k of the Laplace operator with the Dirichlet condition, the series $\sum_k \lambda_k^{-2}$ converges.

3) In this talk we consider the Rayleigh-Stokes problem with a non-local time condition:

$$u(x,T) = \beta u(x,0) + \varphi(x),$$

where β is an arbitrary real number. Note, if $\beta = 0$ then we have the backward problem.

If $\beta = 1$, then we get the following condition

$$u(x,T) = u(x,0) + \varphi(x).$$

It turns out that this non-local problem is well-posed. In other words, a solution to the non-local problem exists and is unique. Moreover, the solution depends continuously on the function $\varphi(x)$ in the non-local condition.

4) The question naturally arises: starting from what value of β does the problem worsen? We will give a comprehensive answer to this question. It turns out that the critical values of the parameter β lie on the half-interval $[0,1)$. If $\beta \notin [0,1)$, then the problem is well-posed according to Hadamard: there is a unique solution and it depends continuously on the data of the problem; if $\beta \in (0,1)$ (the case $\beta = 0$ is the backward problem), then the well-posedness of the problem depends on the location of the eigenvalues of the Laplace operator. For this case, necessary and sufficient conditions are found that guarantee the existence of a solution, but the solution will not be unique.

5) The Rayleigh-Stokes problem involves the fractional derivative ∂_t^α, which describes the behavior of a viscoelastic flow. However, this parameter is often unknown and difficult to measure directly. Therefore, it is undoubtedly interesting to study the inverse problem to determine this physical quantity. We prove that the additional condition $||u(x,t_0)||^2_{L_2(\Omega)} = d_0$ for sufficiently large t_0 uniquely determines the parameter α.

6) Another very important problem studied in this work is to find out the dependence of the behavior of the solution of the initial-boundary value problem on the order of the fractional derivative. In this work, an interesting fact was discovered: if we consider the norm of the solution $||u(x,t_0)||_{L_2(\Omega)}$ as a function of the parameter α, then this is a decreasing function. In other words, the norm acquires its maximum value when the order of the fractional

derivative is close to zero, and its minimum value when this parameter is close to one.

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A note on the delay nonlinear parabolic differential equations

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In the present paper, the initial value problem for the nonlinear delay differential equation

$$\begin{cases} \frac{du}{dt} + Au(t) = f(t, B(t)u(t), B(t)u(t-w)), t \geq 0, \\ u(t) = \varphi(t), -w \leq t \leq 0 \end{cases}$$

in a Banach space E with the strongly positive operator A is considered. Here, $B(t)$ are unbounded operators and $\left\|B(t)A^{-1/2}\right\|_{E\to E} \leq M$. Theorem on the existence and uniqueness of a bounded solution of this problem is established. The application of the main theorem for four different nonlinear parabolic differential equations with time delay is presented. Theorems on the existence and uniqueness of a bounded solution of the initial boundary value problems for four different nonlinear parabolic differential equations with time delay are proved. The first and second-order of accuracy difference schemes for the solution of the one-dimensional nonlinear parabolic equation with time delay are presented. Numerical results are provided.

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Operator approach for the solution of stochastic differential equations

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This is a discuss on the application of the operator approach to stochastic partial differential equations with dependent coefficients. The stability of an abstract Cauchy problem for the solution of a stochastic differential equation in a Hilbert space with the time-dependent positive operator is established. In practice, theorems on stability estimates for the solution of four types of the initial boundary value problems for the one-dimensional and multidimensional stochastic parabolic equation with dependent in t and space variables are proved. Single-step difference schemes generated by exact difference scheme are presented. The main theorems of the convergence of these difference schemes for the approximate solutions of the time-dependent abstract Cauchy problem for the parabolic equations are established. In applications, the convergence estimates for the solution of difference schemes for the approximate solutions for four types of time-dependent stochastic parabolic differential equations are obtained. Numerical results for the $\frac{1}{2}$ and $\frac{3}{2}$ th order of accuracy difference schemes of the approximate solution of mixed problems for stochastic parabolic equations with Dirichlet, Neumann conditions are provided. Numerical results are given.

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On asymptotic properties of eigenvalues of higher order operator-differential equations on the semi-axis

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Let H be a separable Hilbert space. In the space $L_2\left[[0,\infty);H\right]$ we consider the following problem

$$(-1)^n \left(P(x)y^{(n)}\right)^{(n)} + Q(x)y = \lambda R(x)y,\ 0 \le x < \infty \tag{1}$$

$$y(0) =' (0) = ... = y^{(n-1)}(0) = 0 \tag{2}$$

Here $P(x), Q(x), R(x), x \in [0,\infty)$ are the operators that act in all the values in the space H and satisfy the following conditions:

1) the operator function $P(x)$ is n times regular differentiable in the interval and satisfies the condition $(0,\infty) mE \le P(x) \le ME, \quad m; M > 0$ (E is a unit operator in the space H)

2) For arbitrary $x \in [0,\infty)$ $Q(x) \ge \gamma E, \gamma > 0$ $Q^{-1}(x) \in \sigma_\infty$.

3) There exist such numbers $0 < a < \frac{2n+1}{2n}$ A that for all $x \in [0,\infty)$ and $|x-\xi| \le 1$ the inequality $\|[Q(x) - Q(s)]Q^{-a}(x)\| < A\,|x-\xi|$ is valid.

4) There exists such a constant B that for any $x \in [0,\infty)$ and $|x-\xi| > 1$ the inequality

$$\left\| K(\xi) \exp\left(-\frac{1}{2} Im\varepsilon_1 \,|x-\xi|\,\omega\right)\right\| < B$$

is satisfied.

Here $K(x) = P^{-\frac{1}{2}}(x)\ Q(x)P^{-\frac{1}{2}}(x)$

$$\omega(x) = \left\{K(x) + \mu P^{-\frac{1}{2}}(x)R(x)P^{-\frac{1}{2}}(x)\right\}, Im\varepsilon_1 = \min_i \left\{Im\varepsilon_1, \varepsilon_i^{2n} = -1\right\}.$$

5) For all $x, \xi \in [0,+\infty)$

$$\left\|Q^{-1}(x)P^{\pm\frac{1}{2}}(x)Q(x)\right\| < c_1, \left\|Q^{-1}(\xi)P^{\frac{1}{2}}(\xi)P^{\frac{1}{2}}(x)Q(\xi)\right\| < c_2,\ \ c_1, c_2 > 0.$$

6) The operator –function $R(x)$ is determined in the space H for all $x \in [0,\infty)$, has the inverse $R^{-1}(x)$ and satisfies the displacement condition with $P(x), Q(x)$.

7) Denote by $\beta_n(x,s)$ the eigen-values of the operator $R^{-1}(x)P(x)S^{2n} + R^{-1}(x)Q(x)$ arranged in ascending order. Assume that for any positive integer number q the following asymptotic formula condition is valid:

$$\sum_{m=1}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{dxds}{[\beta_m(x,s)+\mu]^{2q}} < \infty$$

The following theorem is valid.

Theorem. *Let conditions 1)-7) be satisfied. Then the following asymptotic formula is valid for the eigenvalues λ_n of the problem (1),(2) as $\mu \to \infty$:*

$$\sum_{m=1}^{\infty}\frac{1}{(\lambda_m+\mu)^{2q}} \approx \frac{1}{2\pi}\sum_{m=1}^{\infty}\int_0^{\infty}\int_0^{\infty}\frac{dxds}{[\beta_m(x,s)+\mu]^{2q}}.$$

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2-local derivations and automorphisms on von Neumann algebras and AW^*-algebras

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Given an algebra A, a linear mapping $T : A \to A$ is called a *homomorphism* (respectively, a derivation) if $T(ab) = T(a)T(b)$ (respectively, $T(ab) = T(a)b + aT(b)$) for a, b in A. A one-to-one homomorphism is called an *automorphism.*

A mapping $\triangle : A \to A$ (not linear in general) is called a *2-local automorphism* (respectively, a *2-local derivation*) on A, if for every x, y in A there exists an automarphism $a_{x,y}$ (respectively, a derivation $d_{x,y}$) on A depending on x and y, such that $\triangle(x) = a_{x,y}(x), \triangle(y) = a_{x,y}(y)$ (respectively, $\triangle(x) = d_{x,y}(x)$ and $\triangle(y) = d_{x,y}(y)$).

The main problem concerning the above notions are to find conditions under which every 2-local automorphism or derivation automatically becomes an automorphism (respectively, a derivation). In the present talk, we give a solution of this problem in the framework of von Neumann algebras and their abstract generalization - AW^*-algebras (Kaplansky algebras).

A coefficient identification problem in a two-dimensional hyperbolic equation with an integral overdetermination condition

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This paper is devoted to the study of the classical solvability of an inverse boundary value problem for a two-dimensional hyperbolic equation with an integral observation. For this purpose, the considered problem is reduced to an auxiliary problem within certain data and the equivalence of the auxiliary problem to the original problem is proved. Then, using the Fourier method, the auxiliary problem is presented in the form of a system of integral equations, and the unique existence of the solution of the obtained system of integral equations for a small value of time is sown. Based on the equivalency of these problems, the existence and uniqueness theorem for the classical solution of the original inverse problem is proved (cf.[1]–[3]).

Formulation of the problem. Let $T > 0$ be a fixed time moment and let $D = Q_{xy} \times \{0 \le t \le T\}$, where Q_{xy} determined by the inequalities $0 < x < 1,\ 0 < y < 1$. In addition, we also assign $D_T := \bar{D}$ and consider the problem of recovering the unknown functions $u(x,y,t) \in C(D_T) \cap C^2(D)$ and $a(t) \in C[0,T]$ that satisfy the following inverse problem

$$u_{tt}(x,y,t) = u_{xx}(x,y,t)+$$

$$+u_{yy}(x,y,t) + a(t)u(x,y,t) + f(x,y,t)\ (x,y,t) \in D_T, \tag{1}$$

$$u(x,y,0) = \varphi(x,y), u_t(x,y,0) = \psi(x,y),\ 0 \le x \le 1,\ 0 \le y \le 1, \tag{2}$$

$$u(0,y,t) = u(1,y,t) = 0,\ 0 \le y \le 1,\ 0 \le t \le T, \tag{3}$$

$$u(x,0,t) = u(x,1,t) = 0,\ 0 \le x \le 1,\ 0 \le t \le T, \tag{4}$$

$$\int_0^1 \int_0^1 w(x,y)u(x,y,t)dxdy = h(t),\ 0 \le t \le T, \tag{5}$$

where the forcing functions $w(x,y)$, $f(x,y,t)$, initial displacement $\phi(x,y)$, initial velocity $\psi(x,y)$, and time-dependent function $h(t)$ are given functions.

Theorem 1. *Suppose that $\varphi(x,y)$, $\psi(x,y)$, $w(x,y) \in C(Q_{xy})$, $f(x,y,t) \in C(D_T)$, $h(t) \in C^2[0,T]$, $h(t) \neq 0$, $0 \leq t \leq T$, and the compatibility conditions*

$$\int_0^1 \int_0^1 w(x,y)\varphi(x,y)dxdy = h(0), \quad \int_0^1 \int_0^1 w(x,y)\psi(x,y)dxdy = h'(0), \tag{6}$$

hold. Then the problem of finding a classical solution of (1)-(5) is equivalent to the problem of determining the functions $u(x,y,t) \in C^2(D_T)$ and $a(t) \in C[0,T]$ satisfying (1)-(4), and the condition

$$h''(t) = \int_0^1 \int_0^1 w(x,y)(u_{xx}(x,y,t) + u_{yy}(x,y,t))dxdy + a(t)h(t) +$$

$$+ \int_0^1 \int_0^1 w(x,y)f(x,y,t)dxdy, \ 0 \leq t \leq T. \tag{7}$$

Existence and uniqueness of the classical solution. We impose the following conditions on the data of problem (1)-(4), (7):

$A_1)$ $\varphi(x,y) \in C^{2,2}(\bar{Q}_{xy}), \varphi_{xxy}(x,y), \varphi_{xyy}(x,y), \varphi_{xxx}(x,y), \varphi_{yyy}(x,y) \in L_2(Q_{xy})$,
$\varphi_x(0,y) = \varphi(1,y) = \varphi_{xx}(0,y) = \varphi_{xx}(1,y) = 0, \ 0 \leq y \leq 1$,
$\varphi(x,0) = \varphi(x,1) = \varphi_{yy}(x,0) = \varphi_{yy}(x,1) = 0, \ 0 \leq x \leq 1$;

$A_2)$ $\psi(x,y) \in C^{1,1}(\bar{Q}_{xy}), \psi_{xx}(x,y), \psi_{yy}(x,y) \in L_2(Q_{xy})$,
$\psi(0,y) = \psi(1,y) = 0, \ 0 \leq y \leq 1$,
$\psi(x,0) = \psi(x,1) = 0, \ 0 \leq x \leq 1$;

$A_3)$ $f(x,y,t) \in C(D_T), \ f_x(x,y,t), f_y(x,y,t) \in L_2(D_T)$,
$f(0,y,t) = f(1,y,t) = 0, \ 0 \leq y \leq 1, \ 0 \leq t \leq T$,
$f(x,0,t) = f(x,1,t) = 0, \ 0 \leq x \leq 1, \ 0 \leq t \leq T$.

$A_4)$ $w(x,y) \in C(\bar{Q}_{xy}), \ h(t) \in C^2[0,T], \ h(t) \neq 0, \ 0 \leq t \leq T$.

The following theorem is valid.

Theorem 2. *Let the conditions $A_1)$ - $A_5)$ be satisfied. Then, problem (1)â€“(3), (7) for the small values of time has a unique solution.*

Thus, from Theorem 1 and Theorem 2, we arrive at the following result.

Theorem 3. *Assume that all the conditions of Theorem 2 are fulfilled and*

$$\int_0^1\int_0^1 w(x,y)\varphi(x,y)dxdy = h(0), \ \int_0^1\int_0^1 w(x,y)\psi(x,y)dxdy = h'(0).$$

Then problem (1)-(5) has a unique for sufficiently small values of T.

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Boundedness of fractional maximal operator in generalized weighted Morrey spaces on Heisenberg groups

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We study Spanne-Guliyev type boundedness of the fractional maximal operator M_α in the generalized weighted Morrey spaces, including weak versions, on Heisenberg groups.

Heisenberg groups, in discrete and continuous versions, appear in many parts of mathematics, including Fourier analysis, several complex variables, geometry, and topology. More detailed information can be found in [2] and the references therein.

Let $\mathbb{H}_n$ be the $2n+1$-dimensional Heisenberg group. That is, $\mathbb{H}_n = \mathbb{C}^n \times \mathbb{R}$, with multiplication $(z,t)\cdot(w,s) = (z+w, t+s+2Im(z\cdot\bar{w}))$, where $z\cdot\bar{w} = \sum\limits_{j=1}^{n} z_j\bar{w_j}$. The inverse element of $u=(z,t)$ is $u^{-1} = (-z,-t)$ and we write the identity of $\mathbb{H}_n$ as $0=(0,0)$. The Heisenberg group is a connected, simply connected nilpotent Lie group. We define one-parameter dilations on $\mathbb{H}_n$, for $r>0$, by $\delta_r(z,t) = (rz, r^2t)$. These dilations are group automorphisms and the Jacobian determinant is r^Q, where $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}_n$. A homogeneous norm on $\mathbb{H}_n$ is given by

$$|(z,t)| = (|z|^2 + |t|)^{1/2}.$$

With this norm, we define the Heisenberg ball centered at $u=(z,t)$ with radius r by $B(u,r) = \{v \in \mathbb{H}_n : |u^{-1}v| < r\}$, and we denote by $B(u,2r) = \{y \in \mathbb{H}_n : \ |u^{-1}2v| < r\}$ the open ball centered at u, with radius $2r$.

Let $f \in Ł_1^{loc}(\mathbb{H}_n)$. The fractional maximal operator M_α is defined by

$$M_\alpha f(u) = \sup_{r>0} |B(u,r)|^{-1+\frac{\alpha}{Q}} \int_{B(u,r)} |f(v)| dV(v), \qquad 0 \le \alpha < Q,$$

where Q is the homogeneous dimension of the Heisenberg group $\mathbb{H}_n$ and $|B(u,r)|$ is the Haar measure of the $\mathbb{H}_n$- ball $B(u,r)$.

By a weight function, briefly weight, we mean a locally integrable function on $\mathbb{H}_n$ which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E)=\int_E w(x)dx$, and denote the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E.

Let $1\le p<\infty$, φ be a positive measurable function on $\mathbb{H}_n\times(0,\infty)$ and w be non-negative measurable function on $\mathbb{H}_n$. We denote by $M_{p,\varphi}(\mathbb{H}_n,w)\equiv M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f\in L^{loc}_{p,w}(\mathbb{H}_n)$ with finite norm

$$\|f\|_{M_{p,\varphi}(w)}=\sup_{u\in\mathbb{H}_n,r>0}\varphi(u,r)^{-1}w(B(u,r))^{-\frac{1}{p}}\|f\|_{L_{p,w}(B(u,r))},$$

where $L_{p,w}(B(u,r))$ denotes the weighted L_p-space of measurable functions f.

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f\in WL^{loc}_{p,w}(\mathbb{H}_n)$ for which

$$\|f\|_{WM_{p,\varphi}(w)}=\sup_{u\in\mathbb{H}_n,r>0}\varphi(u,r)^{-1}w(B(u,r))^{-\frac{1}{p}}\|f\|_{WL_{p,w}(B(u,r))}<\infty,$$

where $WL_{p,w}(B(u,r))$ denotes the weak $L_{p,w}$-space of measurable functions f.

A weight function w is in the Muckenhoupt-Wheeden class $A_{p,q}(\mathbb{H}_n)$, $1<p,q<\infty$, if

$$[w]_{A_{p,q}}=\sup_B\Big(\frac{1}{|B|}\int_B w(u)^q dV(u)\Big)^{1/q}\Big(\frac{1}{|B|}\int_B w(u)^{-p'}dV(u)\Big)^{1/p'}<\infty,$$

where the supremum is taken with respect to all the balls B and $\dfrac{1}{p}+\dfrac{1}{p'}=1$.

While $p=1$, $w\in A_{1,q}$ with $1<q<\infty$ if

$$[w]_{A_{1,q}}=\sup_B\Big(\frac{1}{|B|}\int_B w(u)^q dV(u)\Big)^{\frac{1}{q}}\Big(ess\sup_{u\in B}\frac{1}{w(u)}\Big)<\infty.$$

Let us state the following two main results of the paper. The following Guliyev weighted local estimates are valid (see [3]).

Theorem 1. *Let* $1\le p<q<\infty$, $0\le\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, *and* $\omega\in A_{p,q}(\mathbb{H}_n)$. *Then, for* $p>1$ *the inequality*

$$\|M_\alpha f\|_{L_{q,w^q}(B(u,r))}\lesssim(w^q(B(u,r)))^{\frac{1}{q}}\sup_{t\ge r}\|f\|_{L_{p,w^p}(B(u,t)}\,(w^q(B(u,t)))^{-\frac{1}{q}}$$

holds for any ball $B(u,r)$ and for all $f \in L_{p,w}^{loc}(\mathbb{H}_n)$.

Moreover, for $p=1$ the inequality

$$\|M_\alpha f\|_{WL_{q,w^q}(B(u,r))} \lesssim (w^q(B(u,r)))^{\frac{1}{q}} \sup_{t\geq r} \|f\|_{L_{1,w}(B(u,t)} (w^q(B(u,t)))^{-\frac{1}{q}}$$

holds for any ball $B(u,r)$ and for all $f \in L_{1,w}^{loc}(\mathbb{H}_n)$.

Theorem 2. *Let $1 \leq p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $w \in A_{p,q}(\mathbb{H}_n)$, and (φ_1, φ_2) satisfy the condition*

$$\sup_{t>r} \frac{ess \inf_{t<s<\infty} \varphi_1(u,s)(w^p B((u,s)))^{1/p}}{(w^q(B(u,t)))^{1/q}} \leq C\varphi_2(u,r),$$

where C does not depend on u and r. Then the operator M_α is bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$ for $p>1$ and from $M_{1,\varphi_1}(w)$ to $WM_{q,\varphi_2}(w^q)$ for $p=1$.
Remark. *Note that, in the case $w \equiv 1$, Theorems 1 and 2 were proved in [4]*

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Nasireddin Tusi about the “Beginning of sciences”

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In his books devoted to demonstrative (syllogistic) sciences (“Analysts” (first and second), [1, 2] “Metaphysics”), Aristotle most importantly notes the beginnings of science, to which his extensive reasoning is devoted: “there is a science and there is a beginning , through which we become aware of the definitions” [2. p.262] “it is impossible to prove anything in a circle” [2. p.262]). When interpreting his texts, one can single out the triple structure of the beginnings of science: concepts (or general concepts), axioms and postulates. Aristotle writes: ”That beginning, which is necessary for everyone who will study something, I call an axiom.” “The axioms must be considered by one science, namely that which the philosopher deals with, for these axioms are valid for everything that exists, and not for some special kind apart from all others.” “As for the general beginnings (concepts) for a particular science, this “is a statement of the existence of a separate kind of being with all the characteristic features of the kind, which are studied by this science.” “A postulate is something contrary to the opinion of the student or something that , being proved, is accepted and applied unproved” [2. p.275]

Aristotle introduces other concepts related to the concept, the beginning of science thesis and assumption. All the formulations of Aristotle gave rise to questions and various interpretations among ancient scientists (Proclus , Simplicius and others), as well as among Muslim scientists of the Eastern Middle Ages. (Note that it is currently the subject of discussion in the relevant scientific circles). Proclus [3], for example, in contrast to Aristotle, presented the statement of a postulate as an object of sensory or contemplative experience (“Comments on the ! Book of Euclid”). In his logical treatises “ Asasul Iktibas ” [4] and “ Tajridul mantig ” [5] Tusi gives a practical description of deductive systems, and in the second work he does it laconically, almost in mathematical traditions. Ideologically, Tusi follows Aristotle, but gives his own considerations. The section in [5] “ Scientific provisions ” is devoted to the beginnings of science . He's writing:

”[Science has also] beginnings. They are:

- or premises that do not need mediation: either categorically, as primary principles (axioms) and are called well-known fundamental principles (provisions), or [are used] in this science and are called sources, or fundamental principles (postulates), which are considered in this science from both points of view, however, are clarified in other [sciences]. The student may recognize them, whether he reproves them or is indulgent towards them; or, definitions.

They are all called [basic] provisions ."

Let us note the titles of the paragraphs of this section "Scientific provisions": " subject of science", "questions of science", "methods of applying general principles", "transferring evidence from one science to another (unlike Aristotle, who did not allow this and thereby entered into contradiction with his statement "being provable." If, according to Aristotle, general science is philosophy, then according to Tusi, it is not philosophy , in the Aristotelian sense, but a science, "to which other sciences ascend and are explained in it." In a special way, Tusi notes the concept of "definition", regarding which, as he writes, "I have my own vision." In deriving a definition, "the analysis of the subject is used, in fact, until its highest genera and species differences are achieved by dividing it into parts and particles."

A comparative analysis of the views of Tusi and Aristotle is the subject of further planned work.

Euclid's Elements [2] served as a special material for the discussion and teachings of Aristotle. Various structures and classifications of initial positions in various lists of Euclid, associated either with subsequent corrections of an ideological nature or subsequent insertions, caused different interpretations among researchers (this is reflected in the notes to the translation of D.D. Mordukhai-Boltovsky's "Beginnings"). Numerous adaptations of the "Beginnings" are known in the form of "improvements", "editions", "corrections" of Euclid . among the scientists of the Eastern Middle Ages, among which "Tahriri Uklidis" N.Tusi are of great interest in connection with his logical teachings. Using the logical principles above as a tool, an analysis can be made of the additions that Tusi includes to the "basic principles" in the introduction to the Elements.

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Dances of a non-molicy, orthotropic cylindrical cover in dynamic contact with a bubbled liquid, reinforced with shafts

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The problem of free oscillations of a non-homogeneous, orthotropic, cylindrical cover reinforced with spindles along its thickness, which is in dynamic contact with a bubble liquid, was considered. The solution of the problem was performed based on the application of the Hamilton-Ostrogradsky variation principle. To use the Hamilton-Ostrogradsky variational principle, let us write the total energy of a system consisting of rings, an inhomogeneous cylindrical coating, and a bubbly liquid:

$$J = V + K + \sum_{i=1}^{k_1} (\Pi_i + K_i) + A_0 \tag{1}$$

Here, $\Pi_i,\ K_i$-the potential and kinetic energy of the i- i shaft, A_0- the work done by the pressure force acting on the cylindrical cover by the bubbly liquid in the displacement of the points of the cover in the radial direction, V- the potential, K-kinetic energies of the cylindrical cover, k_1 - the number of shafts. Expressions of mentioned energies are given in [1].

The force acting on the cylindrical cover by the bubbly liquid is as follows [2]:

$$\frac{p}{p_0} = \frac{\alpha_{20}}{\frac{\rho_{0q}}{\rho_q} - \alpha_{10}} \tag{2}$$

To take into account the inhomogeneity along the thickness of the cylindrical coating, it was calculated that Young's modulus and the density of the material are a function of the coordinate varying through the thickness [3].

(1) boundary conditions are also added to the energy expression. In the case of a hinged joint

$$u = v = w = M_x = 0 \quad x = 0; l \tag{3}$$

It is assumed that the coordinate axes coincide with the main curvature lines of the plate, and the shafts are in rigid contact with the coating along these lines:

$$u_i\left(x\right)=u\left(x,y_i\right)+h_i\varphi_1\left(x,y_i\right),\ \ \vartheta_i\left(x\right)=\vartheta\left(x,y_i\right)+h_i\varphi_2\left(x,y_i\right) \quad (4)$$

$$w_i\left(x\right)=w\left(x,y_i\right),\ \ \varphi_i\left(x\right)=\varphi_1\left(x,y_i\right),\ \ \varphi_{kpi}\left(x\right)=\varphi_2\left(x,y_i\right);\ \ h_i=0,5h+H_i^1$$

According to the Hamilton-Ostrogradsky variational principle:

$$\delta W=0$$

Here $W=\int_{t'}^{t''}Jdt$ - is the Hamiltonian effect, $t'$and t'' - are given arbitrary moments of time. Let us find the displacements of the cylindrical cover as follows:

$$\begin{aligned} u&=u_0\cos\frac{\pi mx}{l}\sin\varphi\sin\omega t;\,; \\ \vartheta&=\vartheta_0\sin\frac{\pi mx}{l}\cos n\varphi\sin\omega t; \\ w&=w_0\sin\frac{\pi mx}{l}\sin n\varphi\sin\omega t. \end{aligned} \quad (5)$$

Here u_0,v_0,w_0 are the unknown constants, m,k are the wave numbers in the length and width directions of the cylindrical plate, respectively.

If we make $\rho_0=kw$ p and vary its Jexpression with respect to its u_0,v_0, w_0 independent constants and make the coefficients of independent variations equal to zero, we get a system of homogeneous algebraic equations. If we make its main determinant equal to zero, we get the frequency equation:

$$\det\left\|a_{ij}\right\|=0,\ i,\ j=1,3 \quad (6)$$

The roots of equation (6) were calculated by numerical method. Calculations show that as the value of the concentration parameter increases, the oscillation frequencies increase.

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On the influence of the residual stresses arising from the contact of the cut on the dispersion of axisymmetric longitudinal waves in the two-layer hollow cylinder

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The work deals with the study of the influence of the inhomogeneous residual stresses, caused by the "contact" of the cut, on the dispersion of the axisymmetric longitudinal waves in the two-layer hollow cylinder.

Let us assume that the inner and outer layers of the cylinder occupied the regions $R \le r \le R + h^{(2)}$ and $R + h^{(2)} \le r \le R + h^{(2)} + h^{(1)}$ respectively, in the cylindrical coordinate system $Or\theta z$ associated with the central axis of the cylinder and in the natural state each layer has a longitudinal cut marked by the central angles $\alpha^{(1)}$ and $\alpha^{(2)}$, respectively. Let us assume that $\alpha^{(1)}1$ and $\alpha^{(2)}1$. After removing the cuts by touching their ends, we obtain the hollow cylinders with residual stresses and according to the monograph [1], these residual stresses are calculated by the use of the formulae $\sigma_{(ij)}^{(n)0} = \sigma_{(ij)}^{(n)0}(r, \alpha^{(n)}, E^{(n)}, \nu^{(n)}, h^{(n)}/R)$ (where $(ij) = rr; \theta\theta; zz$ and $n = 1, 2$), the explicit expressions of which are given in the aforementioned monograph. Here and below the values with upper index (2) (index (1)) relates to the inner (outer) cylinder.

We seek to investigate how the foregoing inhomogeneous residual stresses acting in the two-layer hollow cylinder compounded from the aforementioned hollow cylinders and determined by the noted expressions affect the dispersion of axially symmetric longitudinal waves propagating in this two-layer cylinder. Assuming that the materials of the layers are moderately stiff, we perform this investigation in the framework of the second version of the small initial deformation of the linearized 3D theory of elastic waves in bodies with initial stresses [2]. The field equations of this theory can be presented as follows:

$$\frac{\partial t_{rr}^{(m)}}{\partial r} + \frac{\partial t_{zr}^{(m)}}{\partial z} + \frac{1}{r}(t_{rr}^{(m)} - t_{\theta\theta}^{(m)}) =$$

$$= \rho^{(m)}\frac{\partial^2 u_r^{(m)}}{\partial t^2}, \ldots, t_{rr}^{(m)} = \sigma_{rr}^{(m)} + \sigma_{rr}^{(m)0}(r)\frac{\partial u_r^{(m)}}{\partial r}, \ldots,$$

$$\sigma_{(ij)}^{(m)} = \lambda^{(m)}(\varepsilon_{rr}^{(m)}+\varepsilon_{\theta\theta}^{(m)}+\varepsilon_{zz}^{(m)})+2\mu^{(n)}\varepsilon_{(ij)}^{(m)}, \ldots, \varepsilon_{rz}^{(m)} = \frac{1}{2}\left(\frac{\partial u_r^{(m)}}{\partial z} + \frac{\partial u_z^{(m)}}{\partial r}\right), \; m = 1,2 \quad (1)$$

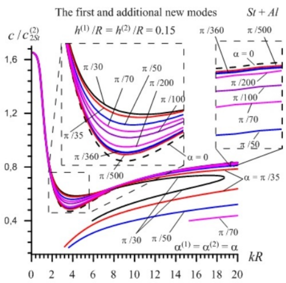
In (1) and above the conventional notation is used. We add to equations in (1) the corresponding boundary and contact conditions which can be presented as follows.

$$t_{rr}^{(1)}\Big|_{r=R+h^{(1)}+h^{(2)}} = 0, \ldots, t_{rr}^{(1)}\Big|_{r=R+h^{(1)}} = t_{rr}^{(2)}\Big|_{r=R+h^{(1)}}, \ldots, t_{rz}^{(2)}\Big|_{r=R} = 0. \quad (2)$$

For the solution to the formulated problem we attempt to employ the discrete-analytical method developed and employed in the paper [3], according to which, the intervals $[R, R+h^{(2)}]$ and $[R+h^{(2)}, R+h^{(1)}+h^{(1)}]$ are divided into N_2 and N_1 numbers of subintervals or sublayers, respectively. The thickness of the sublayers of the region is equal to $h^{(2)}/N_2$ and in the $n_2 - th$ sub-layer, the relation $(R+(n_2-1)h^{(2)}/N_2) \leq r \leq (R+n_2h^{(2)}/N_2)$ takes place, where $1 \leq n_2 \leq N_2$. We can also conclude that the thickness of the sublayers of the region $[R+h^{(2)}, R+h^{(2)}+h^{(2)}]$ is equal to $h^{(1)}/N_1$ and in the $n_1 - th$ sub-layer, the relation $(R+h^{(2)}+(n_1-1)h^{(1)}/N_1) \leq r \leq (R+h^{(2)}+n_1h^{(1)}/N_1)$ takes place, where $1 \leq n_1 \leq N_1$.It is assumed that within each of the foregoing sublayers, the residual stresses are taken as constants. Within these frameworks, we succeed to find analytical solutions to the governing field equations in (1) for each sublayer that contain unknown constants. Satisfying the complete contact conditions between the sublayers we obtain the system of homogeneous linear algebraic equations with respect to the mentioned constants. According to the usual procedure, we obtain the dispersion equation from this system of equations. The dispersion equation is solved numerically by employing the corresponding PC program in MATLAB.

Numerical results are obtained for the case where the materials of the layers are steel (St) and aluminum (Al) and, according to the monograph [2], the material density, modulus of elasticity, Poisson's coefficients and shear wave propagation velocity of the St (Al) we select as $\rho_{St} = 7795\,kg/m^3$, $E_{St} = 19.6GPa$, ν_{St} and $c_{2St} = 3152m/s, \rho_{Al} = 2770\,kg/m^3$, $E_{Al} = 7.28\,GPa$, $\nu_{Al} = 0.3$ and $c_{2Al} = 3179m/s$), respectively. Assume that $\alpha^{(1)} = \alpha^{(2)} = \alpha$ and consider some numerical results obtained for the case where the inner and outer layers are made of steel and aluminum, respectively (denote this case as

$(St\,+\,Al)$). Consider the case where $h^{(1)}/R = h^{(2)}/R = 0.15$ for which the dispersion curves are presented in Fig. 1



Fiq. 1.

From the results presented in Fig. 1, it is clear that additional new types of dispersion curves appear and these curves approach the dispersion curves of the first mode with increasing kR. To distinguish these additional modes related to the same angle, we introduce the term "maximum additional mode". This means that the wave propagation velocity in this additional mode is larger than the corresponding velocity in the other additional modes constructed for the same angle

At the same time, these results show that there are cases where this approaching is completed with a contact of the dispersion curves of the first mode with the dispersion curve of the "maximum additional mode".

Besides all of these, in the work, the many other numerical results on the influence of the inhomogeneous residual stresses on the dispersion curves constructed for various problem parameters and these results are discussed.

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The absence of positive global periodic solution of a second-order semi-linear parabolic equation with time-periodic coefficients

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Denote $B_R = \{x : |x| < R\}$, $B'_R = \{x : |x| > R\}$, $B_{R_1,R_2} = \{x : R_1 < |x| < R_2\}, Q_T^{R_1,R_2} = B_{R_1,R_2} \times (0,T)$, $Q_T^{R,\infty} = B'_R \times (0,T)$, $Q_T = \Omega \times (0,T)$, $Q = \Omega \times (-\infty; +\infty)$, where Ω is the exterior of a compact set D in R_x^n where D contains the origin.

Consider the equation

$$\frac{\partial u}{\partial t} = div(A(x,t)\nabla u) + h(x,t,u) \tag{1}$$

in the cylinder Q, where $A(x,t) = (a_{ij}(x,t))_{ij=1}^n$, $h(x,t,u) : \Omega \times (-\infty, +\infty) \times [0, +\infty) \to R$, $a_{ij}(x,t)$ are bounded, measurable, T- periodic in t functions, and there exist constants ν_1, ν_2 such that

$$\nu_1 |\xi|^2 \le (A\xi, \xi) \le \nu_2 |\xi|^2 \tag{2}$$

for every $(x,t) \in Q, \xi = (\xi_1, ..., \xi_n) \in R^n$.

Here $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$, $A\nabla u = \left(\sum\limits_{j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\right)_{i=1}^n$, $(A\xi, \eta) = \sum\limits_{i,j=1}^n a_{ij}\xi_i\eta_j$, $\xi = (\xi_1, ..., \xi_n)$, $\eta = (\eta_1 ...., \eta_n)$, $div(A\nabla u) = \sum\limits_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x,t)\frac{\partial u}{\partial x_j})$.

We will study the existence of a global positive solution of equation (1). Before giving a definition for the solution, we consider the following function space:

$$W_2^{1,1/2}(Q_T) = \{u(x,t+T) = u(x,t), u(x,t) \in W_2^{1,0}(Q_T), \sum_{k=-\infty}^{\infty} |k| \int_\Omega |u_k(x)|^2 dx < \infty\},$$

where

$$u_k(x) = \frac{1}{T}\int_0^T u(x,t)\exp\left\{-ik\frac{2\pi}{T}t\right\}dt.$$

The norm in this space is defined as follows.

$$\|u\|^2_{W_2^{1,\,1/2}(Q_T)} = \|u\|^2_{L_2(Q_T)} + \|\nabla u\|^2_{L_2(Q_T)} + \sum_{k=-\infty}^{\infty}|k|\int_\Omega |u_k(x)|^2\,dx.$$

By $\overset{\circ}{W}{}_2^{1,1/2}(Q_T)$ we mean a completion of $C^{0,\infty}(Q_T)$ with respect to the norm $\|\cdot\|_{W_2^{1,\,1/2}(Q_T)}$, where $C^{0,\infty}(Q_T)$ is a set of infinitely differentiable functions on Q , which are T periodic in t and vanish in the vicinity of $\partial\Omega$.

A solution of equation (1) is defined as a function $u(x,t) \in W^{1,\,1/2}_{2,loc}(Q_T) \cap L_{\infty,loc}(Q_T)$ satisfying the corresponding integral identity.

In the presented article the nonlinearity has a more general form, namely, we assume that $h(x,t,u) \geq \tilde{h}(x,u) \geq 0$ for all $(x,t) \in Q$ and $\tilde{h} : \Omega \times \bar{R}^+ \to R^+$ is a function satisfying the following.

(H): (a) for any $x \in B'_e$

$$\frac{\tilde{h}(x,s_1)}{s_1} \geq \frac{\tilde{h}(x,s_2)}{s_2} \text{ if } s_1 \geq s_2 > 0$$

(b) for any $\tau > 0$,

$$\lim_{|x|\to+\infty}\inf \tilde{h}(x,\tau\left|x^{2-n}\right|)\,|x|^n > C_0$$

If (b) fails, we assume that

(b1) there exists $\sigma_1 \in (0,1)$ such that for any $\tau > 0$,

$$\lim_{|x|\to+\infty}\inf \tilde{h}(x,\tau\left|x^{2-n}\right|)\,|x|^n\,(\ln|x|)^{\sigma_1} > 0;$$

(b2) there exists $\sigma_2 > 0$ such that for any $\tau > 0$,

$$\lim_{|x|\to+\infty}\inf\frac{\tilde{h}(x,\tau\,|x|^{2-n}\,(\ln|x|)^{\sigma_2})}{\tau\,|x|^{-n}\,(\ln|x|)^{\sigma_2}} > C_0$$

Denote

$$L_0 \equiv div(A(x,t)\nabla u) - \frac{\partial u}{\partial t}.$$

Consider the linear inequality

$$L_0 u + P(x,t)u \leq 0 \text{ in } Q_T^{R,\infty}, \tag{3}$$

where

$$P(x,t+T) = P(x,t), P(x,t) \in L_{\infty,loc}(Q_T^{R,\infty}).$$

Lemma 1. *There exists a constant $C_0 > 0$ depending on n, λ_1, λ_2 and not depending on R such that, if $P(x,t) \geq \frac{C_0}{|x|^2}$, then inequality (3) has no positive solution in $Q_T^{R,\infty}$.*

Note that when $a_{ij}(x,t) = \delta_{ij}$, then $C_0 = \left(\frac{n-2}{2}\right)^2$.

Lemma 2. *Let $n \geq 3$ and $u(x,t) \in W_{2,loc}^{1,\,1/2}(Q_T^{R,\infty})$ be a function continuous and nonnegative in $\bar{Q}_T^{R,\infty}$ and such that $L_0 u \leq 0$ in $Q_T^{R,\infty}$ and $u(x,t) > 0$ on S_R. Then $u(x,t) \geq \beta_0 \left|x\right|^{2-n}, (x,t) \in Q_T^{R,\infty}, \beta_0 = const > 0$.*

The main result is the following theorem.

Theorem. *Let $n \geq 3$, $A(x,t)$ satisfies the condition (2) . Then under the assumption (H) equation (1) has no super solution in Q.*

On weak compactness in weighted variable Lebesgue spaces

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In this abstract, we study the weak compactness of subsets of weighted variable Lebesgue spaces. We also study a L-weakly compact and weakly compact inclusions between weighted variable Lebesgue spaces are studied.

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The asymptotic formulas for the sum of squares of negative eigenvalues of a differential operator with operator coefficient

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Let H be a separable Hilbert space. In separable Hilbert space $L_2(0,\infty;H)$ we consider the operator L defined by the differential expression

$$l(y) = -(p(x)y'(x))' - Q(x)y(x)$$

and with the boundary condition

$$y(0) = 0.$$

Suppose that the scalar function $p(x)$ and the operator function $Q(x)$ satisfy the following conditions:

1) There are constants c_1 and c_2 such that $0 < c_1 \leq p(x) \leq c_2$.

2) $p(x)$ is a continuous, non-decreasing function and it has a bounded derivative on $[0,\infty)$.

3) $Q(x) : H \to H$ is an absolutely continuous, self-adjoint and positive operator, for all $x \in [0,\infty)$.

4) $Q(x)$ is monotonous decreasing.

5) $Q(x)$ is continuous with respect to the norm on the space $B(H)$ and $\lim\limits_{x\to\infty} \|Q(x)\| = 0$.

It is known that the operator L is self-adjoint, semi-bounded below and the negative part of its spectrum is discrete [1].

Let $\lambda_1 \leq \lambda_2 \leq \ldots \lambda_n \leq \ldots$ be negative eigenvalues of the operator L. In this work, we have found some asymptotic formulas for $\sum\limits_{\lambda_j < -\varepsilon} \lambda_j^2$ $(\varepsilon > 0)$, as $\varepsilon \to 0$.

Let $\alpha_1(x) \geq \alpha_2(x) \geq \ldots \geq \alpha_j(x) \geq \ldots$ be eigenvalues of the operator $Q(x) : H \to H$.

Since $Q(x) > 0$ for every $x \in [0, \infty)$, we have $\alpha_j(x) > 0$ $(j = 1, 2, \ldots)$.

Since, $Q(x) > 0$ is continuous, the function $\alpha_1(x)$ is continuous in $x \in [0, \infty)$. Since, $Q(x)$ monotonous decreasing, it can be shown that the functions $\alpha_1(x), \alpha_2(x), \ldots, \alpha_j(x), \ldots$ are monotonous decreasing.

We suppose that the function $\alpha_1(x)$ satisfies following condition:

6) For every $\eta > 0$

$$\lim_{x\to\infty} \alpha_1(x) x^{a_0-\eta} = \lim_{x\to\infty} [\alpha_1(x) x^{a_0+\eta}]^{-1} = 0$$

where a_0 is a constant which belongs to the interval $(0, \frac{2}{3})$.

The following theorem holds.

Theorem 1. *We suppose that the conditions 1)-6) are satisfied. In addition, if the series*

$$\sum_{j=1}^{\infty} (\alpha_j(0))^m$$

is convergent for a fixed number m which satisfies the condition

$$0 < m < \frac{(2-3a_0)^2}{2a_0(4-3a_0)},$$

then the asymptotic formula

$$\sum_{\lambda_j<-\varepsilon} \lambda_j^2 = \frac{1}{15\pi}[1 + O(\varepsilon^{t_0})] \sum_j \int\limits_{\alpha_j(x)\geq\varepsilon} \sqrt{\frac{\alpha_j(x)-\varepsilon}{p(x)}} [8\alpha_j^2(x) + 4\alpha_j(x)\varepsilon + 3\varepsilon^2] dx$$

is satisfied when $\varepsilon \to 0$. Here t_0 is a positive constant.

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On existence theorem on solvability in the small in Weighted Grand sobolev spaces

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We need some necessary standard notations and facts. Z_+ will be the set of non-negative integers. $B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$ will denote the open ball in R^n centered at x_0, where $|x| = \sqrt{x_1^2 + ... + x_n^2}$, $x = (x_1, ..., x_n)$. $\Omega_r(x_0) = \Omega \bigcap B_r(x_0)$, $B_r = B_r(0)$, $\Omega_r = \Omega_r(0)$. $|M|$ will stand for the Lebesgue measure of the set M; $\partial\Omega$ will be the boundary of the domain Ω; $\bar{\Omega} = \Omega \bigcup \partial\Omega$; $diam\,\Omega$ will stand for the diameter of the set Ω; For $\forall \varepsilon \in (0, q-1)$ we will denote $q_\varepsilon = q - \varepsilon$. q' is conjugate to q number: $\frac{1}{q} + \frac{1}{q'} = 1$.

Let $\Omega \subset R^n$ be some bounded domain with the rectifiable boundary $\partial\Omega$. We will use the notation of [1]. $\alpha = (\alpha_1, ..., \alpha_n)$ will be the multiindex with the coordinates $\alpha_k \in Z_+$, $\forall k = \overline{1,n}$; $\partial_i = \frac{\partial}{x_i}$ will denote the differentiation operator, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} ... \partial_n^{\alpha_n}$. For every $\xi = (\xi_1, ..., \xi_n)$ we assume $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} ... \xi_n^{\alpha_n}$. Let L be an elliptic differential operator of m-th order

$$L = \sum_{|p| \le m} a_p(x)\, \partial^p,$$

where $p = (p_1, ..., p_n)$, $p_k \in Z_+$, $\forall k = \overline{1,n}$, $a_p(\cdot) \in L_\infty(\Omega)$ are real functions.

In what follows, by solution of the equation $Lu = f$ we mean a strong solution.

Let us define the grand Lebesgue space $L_{q)}(\Omega)$. Grand Lebesgue space $L_{q)}(\Omega)$, $1 < q < +\infty$, where $\Omega \subset R^n$ – bounded domain, is a Banach space of (Lebesgue) measurable functions f on Ω with norm

$$\|f\|_{L_{q)}(\Omega)} = \sup_{0<\varepsilon<q-1} \left(\varepsilon \int_\Omega |f|^{q-\varepsilon}\, dx \right)^{\frac{1}{q-\varepsilon}} .$$

Let $\rho : \Omega \to [0, +\infty]$ be a weight function, i.e. ρ – measurable and $|\rho^{-1}\{0; +\infty\}| =$

0. The weighted case of $L_{q)}(\Omega)$ we define by norm

$$\|f\|_{L_{q),\rho}(\Omega)} = \sup_{0<\varepsilon<q-1} \left(\varepsilon \int_{\Omega} |f|^{q-\varepsilon} \rho dx\right)^{\frac{1}{q-\varepsilon}},$$

and corresponding space is denoted by $L_{q),\rho}(\Omega)$. Also the corresponding Sobolev space $W^m_{q),\rho}(\Omega)$ is defined by norm

$$\|f\|_{W^m_{q),\rho}(\Omega)} = \sum_{|\alpha|\le m} \|\partial^\alpha f\|_{L_{q),\rho}(\Omega)}.$$

We will assume that every function defined on Ω is extended by zero to $R^n\backslash\overline{\Omega}$. Let T_δ be a shift operator, i.e. $(T_\delta f)(x) = f(x+\delta)$, $\forall x \in \Omega$, where $\delta \in R^n$ is an arbitrary vector. Let

$$_sL_{q),\rho}(\Omega) = \left\{f \in L_{q),\rho}(\Omega) : \|T_\delta f - f\|_{L_{q),\rho}(\Omega)} \to 0,\ \delta \to 0\right\}.$$

The following separable weighted Sobolev space is also defined

$$_sW^m_{q)}(\Omega) = \left\{f \in W^m_{q)}(\Omega) : \|T_\delta f - f\|_{W^m_{q)}(\Omega)} \to 0,\ \delta \to 0\right\}.$$

In the following we need some facts about boundedness of convolution in weighted Lebesgue spaces. Let

$$\|f\|_{L_{p,\rho}(\Omega)} = \left(\int_{\Omega} |f(x)^p| \rho(x)\, dx\right)^{\frac{1}{p}},\ 1 \le p < \infty.$$

Corresponding weighted Lebesgue space is denoted by $L_{p,\rho}(\Omega)$.

It is valid the following

Lemma 1. *Let* $\rho(x) = |x|^{\alpha q}$, $A_{n;q} = \max\left\{-\frac{n}{q}; -n\left(1-\frac{1}{q-\varepsilon_0}\right)+\delta_0\right\}$, *where* $\varepsilon_0 = \frac{\delta_0 q^2}{n+\delta_0 q}$ *and* $\delta > 0$ *be sufficiently small number. If it holds*

$$A_{n;q} < \alpha \le n\left(1-\frac{1}{q-\varepsilon_0}\right) - \delta_0,$$

then for $\forall r_0 > 0$, *there exists* $C_{r_0} > 0 : \forall r \in (0, r_0)$, *such that it is valid*

$$\|I(\cdot)\|_{L_{q),\rho}(r)} \le C_{r_0} r^{m-|p|+\delta_0} \|\psi\|_{L_{q),\rho}(r)},$$

where $C_{r_0} = (q-1)^{\frac{1}{q}} \left(\int_{B_{r_0}} \rho dx \right)^{\frac{q-1}{q^2}}$.

In ${}_sW^m_{q),\rho}(\Omega)$ along with norm $\|\cdot\|_{W^m_{q),\rho}(\Omega)}$ let's consider the following norm

$$\|f\|_{N^m_{q),\rho}(\Omega)} = \sum_{|p|\le m} d_\Omega^{|p|-\frac{n}{q}} \|\partial^p f\|_{L_{q),\rho}(\Omega)},$$

where $d_\Omega = diam\ \Omega$ and the corresponding space we will denote by $N^m_{q),\rho}(\Omega)$. Accept $N^0_{q),\rho}(\Omega) = N_{q),\rho}(\Omega)$. It is not difficult to see that the norms $\|\cdot\|_{W^m_{q),\rho}(\Omega)}$ and $\|\cdot\|_{N^m_{q),\rho}(\Omega)}$ are equivalent, and therefore the collection of functions of spaces $W^m_{q),\rho}(\Omega)$ and $N^m_{q),\rho}(\Omega)$ coincide. In addition, assume

$${}_sN^m_{q),\rho}(\Omega) = \left\{ f \in N^m_{q,\rho}(\Omega) : \|T_\delta f - f\|_{N^m_{q,\rho}(\Omega)} \to 0, \delta \to 0 \right\}.$$

Accept the following

Definition 1. *We will say that the operator L has the property $P_{x_0})$ if its coefficients satisfy the conditions: i) $a_p \in L_\infty(B_r(x_0))$, $\forall |p| \le m$, for some $r > 0$; ii) $\exists r > 0 :$ for $|p| = m$ the coefficient $a_p(\cdot)$ coincides a.e. in $B_r(x_0)$ with some function bounded and continuous at the point $x_0 \in \Omega$.*

Analogously to the work [2] we prove the following existence theorem on solvability in the small.

Theorem 1. *Let L be a m–th order elliptic operator which has the property $P_{x_0})$ at some point $x_0 \in \Omega$ and weight $\rho(x) = |x|^{\alpha q}$, $1 < q < +\infty$, satisfy all conditions of Lemma 1. For $\forall f \in {}_sN_{q),\rho}(\Omega)$ for sufficiently small $r > 0$, there exists a solution of the equation $Lu = f$ belonging to the class $N_{q),\rho}(B_r(x_0))$.*

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Numerical modeling of multi-phase filtration in a deformable porous medium

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The flow of a viscous fluid in a porous deformable medium has many practical applications in various fields [1]. Early studies considered the flow of fluid through compacted areas of sand [2], [3]. Later, these works were expanded by taking into account the anisotropy of a compressible medium [4] and the consideration of wave propagation in a deformable medium [1].

The paper considers the problem of numerical modeling of the process of multiphase filtration in a deformable porous medium, taking into account external influences. A mathematical model and a numerical algorithm for its implementation have been developed. The effect of medium deformation on the process of multiphase filtration has been studied.

Consider an oil reservoir opened by two production wells. Let us assume that oil and water phases are involved in the filtration process in a deformable porous medium. The mathematical model of two-phase filtration in a deformable porous medium has the following form:

Law of conservation of mass for the oil phase

$$\frac{\partial}{\partial t}(m\rho_o s_o) + \frac{\partial}{\partial x}(\rho_o u_o) = 0, \tag{1}$$

where s_o- saturation of the pore space with oil, ρ_o- oil phase density, m– porosity, u_o– oil phase velocity.

Law of conservation of mass for the water phase

$$\frac{\partial}{\partial t}(m\rho_w s_w) + \frac{\partial}{\partial x}(\rho_w u_w) = 0 \tag{2}$$

where s_w- saturation of the pore space with water, ρ_w- water phase density, u_w – water phase velocity.

We put $\rho_o = const$, $\rho_w = const$ and rewrite the equation in the following form.

$$\frac{\partial}{\partial t}(m s_o) + \frac{\partial}{\partial x}(u_o) = 0, \tag{3}$$

$$\frac{\partial}{\partial t}(ms_w) + \frac{\partial}{\partial x}(u_w) = 0, \tag{4}$$

Phase filtration velocities according to Darcy's law [4]

$$u_o = -\frac{Kk_o}{\mu_o}\frac{\partial p}{\partial x}, \quad u_w = -\frac{Kk_w}{\mu_w}\frac{\partial p}{\partial x}, \tag{5}$$

where $K,\ k_o,\ k_w$– absolute and relative phase permeability coefficients, μ_o- oil viscosity, μ_w- water viscosity, p – pressure.

Change in porosity [4]

$$m = m_0 + \beta_M (p - p_0), \tag{6}$$

where p_0 – fixed pressure, m_0– porosity coefficient at $p = p_0$, β_m– compression coefficient of reservoir.

Change in absolute permeability [4]

$$K = K_0 (1 - a_K (p_0 - p)), \tag{7}$$

where a_K– permeability coefficient, K_0– initial absolute permeability

Let us add to the system of equations the obvious equality

$$s_o + s_w = 1, \tag{8}$$

and dependences of relative phase permeability coefficients on saturations

$$k_o = \begin{cases} \left(\frac{0{,}85 - s_w}{0{,}8}\right)^{2{,}8} \cdot (1 + 2,4 \cdot s_w), & 0 \le s_w \le 0,85, \\ 0 & 0,85 \le s_w \le 1; \end{cases} \tag{9}$$

$$k_w = \begin{cases} 0 & 0 \le s_w \le 0,85, \\ \left(\frac{s_w - 0{,}2}{0{,}8}\right)^{3{,}5}, & 0,85 \le s_w \le 1; \end{cases} \tag{10}$$

To derive the equation for pressure, we sum equations using and and assume $\mu_o = const, \quad \mu_w = const$. As a result, we get

$$\beta_M \frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left[\left(\frac{Kk_0}{\mu_0} + \frac{Kk_w}{\mu_w}\right)\frac{\partial p}{\partial x}\right] \tag{11}$$

By setting the initial

$$p(x,0) = p^0, \quad s_o(x,0) = s_o^0, \quad s_w(x,0) = s_w^0 \tag{12}$$

and boundary conditions

$$p\,(0,t) = p_0,\ \ p\,(l_1,t) = p_1^*,\quad p\,(l_2,t) = p_2^*,\quad p\,(L,t) = p_0, \tag{13}$$

$$s_o\,(0,t) = s_o^* s_w\,(0,t) = s_w^*,\quad s_o\,(L,t) = s_o^{**},\quad s_w\,(L,t) = s_w^{**}, \tag{14}$$

where l_1– coordinate of the first well, l_2– second well coordinate, L– filtration area length, p_1^*– bottom hole pressure of the first well, p_2^*– bottom hole pressure of the second well, s_o^* and s_w^* – oil saturation and water saturation on the left boundary, s_o^{**} and s_w^{**}– oil saturation and water saturation on the right boundary.

Problem was solved by the large particle method. Taken results show that an increase in the value of the compression coefficient leads to a slower pressure drop both in the entire reservoir and in the inter-well zone. The difference in pressure drop in wells significantly affects the filtration process both in the entire reservoir and in the inter-well zone.

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On some properties of the Lorentz Gegenbauer spaces

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In the paper we introduce Lorentz type space associated with Gegenbauner differential operator G_λ

$$G_\lambda = \left(x^2-1\right)^{\frac{1}{2}-\lambda} \frac{d}{dx} \left(x^2-1\right)^{\lambda+\frac{1}{2}} \frac{d}{dx}, x \in (0,\infty), \lambda \in \left(0, \frac{1}{2}\right).$$

For this spaces the emedding theorems is proved. Moreover, is proved that this space is a Banach space.

On the representing of functions of several variables by multidimensional J-fractions with independent variables

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We consider the problem of representing of the functions of several variables by multidimensional J-fractions with independent variables. A generalization of Gragg's classical algorithm (see, [3]) is constructed, which allows us to calculate the coefficients of the corresponding multidimensional J-fraction with independent variables by the coefficients of a given formal multiple Lourent series.

As an example, the function of two variables

$$\begin{aligned}\Psi_1(z_1,z_2)&=\psi_1(z_1)+\psi_1(z_2+\psi_1(z_1))\\ &=\int_0^\infty\frac{te^{-tz_1}}{1-e^{-t}}dt+\int_0^\infty\frac{s}{1-e^{-s}}\exp\left\{-sz_2-\int_0^\infty\frac{ste^{-tz_1}}{1-e^{-t}}dt\right\}ds,\end{aligned}$$

where $\psi_1(.)$ is trigamma function (see [1, p. 260]), has the asymptotic representation as a formal double Laurent series

$$\Psi_1(z_1,z_2)\approx\sum_{k=0}^{\infty}\frac{B_k^+}{z_1^{k+1}}+\sum_{r=0}^{\infty}\frac{B_r^+}{z_2^{r+1}}\left(\sum_{s=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{-B_k^+}{z_1^{k+1}z_2}\right)^s\right)^{r+1},$$

where $z_i\to\infty$, $|\arg(z_i)|<\pi$, $i=1,2$,

$$B_k^+=1-\sum_{r=0}^{k-1}\binom{k}{r}\frac{B_r^+}{k-r+1},\quad k\ge 0,$$

are the Bernoulli numbers. According to [2, Theorem 3], using the generalization of classical Gragg's algorithm, for $\Psi_1(z_1,z_2)$ we obtained the corresponding two-dimensional J-fraction with independent variables

$$\sum_{i_1=1}^{2}\frac{p_{e_{i(1)}}}{q_{e_{i(1)}}+z_{i_1}}+\sum_{i_2=1}^{i_1}\frac{p_{e_{i(2)}}}{q_{e_{i(2)}}+z_{i_2}}+\sum_{i_3=1}^{i_2}\frac{p_{e_{i(3)}}}{q_{e_{i(3)}}+z_{i_3}}+\dots,$$

where

$$p_{e_1+re_2} = p_{e_2} = 1, \quad r \geq 0,$$
$$p_{ke_1+re_2} = p_{ke_2} = \frac{(k-1)^4}{4(2k-3)(2k-1)}, \quad k \geq 2,\ r \geq 0,$$
$$q_{ke_1+re_2} = q_{ke_2} = -1/2, \quad k \geq 1,\ r \geq 0.$$

Numerical experiments confirm the expediency and effectiveness of using multidimensional J-fractions with independent variables as an approximation tool. Nevertheless, the problems of improving and developing new methods of studying their convergence remain open.

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The place and tasks of teaching the subject of mathematical analysis in the training of a mathematics teacher

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In all situations related to education, the following questions are traditionally on the agenda. Why study? What to learn? How to learn?

In order to answer these questions, all the sciences related to education investigate the purpose, content and methodology of education, improve and develop it according to the requirements of the time. The purpose, content and methodology of mathematics education, which is an inseparable and perhaps the most important direction of education, is the object of research of mathematical scientists and pedagogues. Education is society's order to school. A developed society makes higher demands on education and ensures the rise of the scientific and technical level of society in higher education. If we do not take into account the exceptions related to individual individuals, it is impossible to achieve the progress of the modern world level of scientific and technical science-intensive economy in a country with an imperfect education system. On the other hand, in countries with a weak science-intensive economy, high indicators are not achieved in education. Therefore, in order to prepare the optimal system of specialist training, first of all, the goal of education, the content that will ensure the achievement of this goal, and the forms of mastering the content should be known.

In this sense, the purpose of mathematics teacher training is determined by the purpose of general education by meeting the requirements of scientific and technical progress. This goal should take into account school reforms while reconciling with the social order. A graduate of a pedagogical university should know the tasks facing school mathematics and be able to fulfill them as a teacher. This will be possible when the mathematical subjects taught in higher school are inextricably linked with secondary school mathematics, and their life can be justified by necessity.

The full implementation of the goal of secondary education is an indicator of the high level of specialized professional education in pedagogical universities. The level of mathematical training of the graduates of the pedagogical university is the main factor that ensures the mathematical training of their students. The well-known methodist G. Freudenthal puts forward 4 minimum requirements for the theoretical and mathematical preparation of the future teacher. He believes that higher education should enable him to use the fundamental methods of modern mathematics, gain fundamental knowledge to understand the structure of modern mathematics, understand the applications of mathematics, and have a certain idea of conducting mathematical research.

A mathematics teacher must have sufficient scientific and pedagogical training to build a mathematical model of real events. A graduate of a pedagogical higher school should be able to express many physical, technical, economic issues with abstract mathematical symbols, and should have a mathematical apparatus to study a known or constructed mathematical model. He should be able to apply the learned theoretical issues in solving practical issues. Unlike other university graduates, the graduates of the pedagogical university should have the ability to transfer these qualities to their students and make them love these values. A teacher of mathematics should be able to cultivate a stable interest in mathematics, mathematical intuition, and create motivation for the subject he is teaching.

For this purpose, the application of mathematics should be able to show the beauty of mathematics with facts related to its historical development. All this will ultimately affect the development of the graduate's understanding, mathematical culture, scientific outlook, and the development of pedagogical skills. Most of the subjects in the mathematics teacher training curricula contribute to the development of the mentioned competencies in graduates. Mathematical analysis bears a great responsibility in the formation of these competencies. The report focuses on some points related to the teaching of this subject.

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On some imbedding theorems in the space $S^{r,\chi}_{p,\theta}(R^n : E)$

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Let E be a Banach space, $p = (p_1, ..., p_n)$, $r = (r_1, ..., r_n)$, $r_j \geq 0$, e_r-carrier of the vector r. We consider the $SB^{r,\chi}_{p,q}(R^n : E)$ space of functions $f \in L_p(R^n : E)$, which has derivatives D^{k^e} of order k^e for all $e \subset e_r$ and for them has the relation

$$\sum_{e^1+e^2=e} \Big\{ \prod_{j\in e^r} \delta_j \Big[\int \cdots \int_{0\leq t_j\leq\delta_j, j\in e^1} \cdots \int_{\delta_j\leq t_j\leq 2, j\in e^2} \bigcap_{j\in e^1} t^{-q(r_j-k_j)-1} \bigcap_{j\in e^2} t^{-q(r_j-k_j+1)-1}$$

$$\Omega^{m^e}\Big(D^{k^e} f : t^e\Big)_{L_p(R^n:E)}\Big]^q dt_1...dt_n \Big\}^{\frac{1}{q}} \leq M^{r^e}_{p,q} \prod_{j\in e}^{n} \varphi_j(\delta^{e^j})$$

where the $\sum\limits_{e^1+e^2=e}$ sum is extended to all possible subsets $e^1, e^2 \subset e \subset e_r$ for which $e^1 \cap e^2 = \emptyset$, $e^1 + e^2 = e$ for $\forall e \subset e_r$, $\varphi_i(\delta_j)$ is a continuous non-negative function $\chi(\delta_j) = o(\delta_j)$, $m = (m_1, ..., m_n)$, $r = (r_1, ..., r_n)$, $k = (k_1, ..., k_n)$, $m_j > r_j - k_j$, $j = 1, ..., n$. $SB^r_{p,q}(R^n : E)$ is a complete normed space.

Define the norm

$$\|f; S^{r,\chi}_{p,\theta} B(R^n : E)\| = \sup_{0\leq\delta_j\leq 2, j\in e_r} \sum_{e\subset e_r} M^r_{p,\theta}$$

Theorem 1. *Let $e_r = e_m$, $1 \leq m \leq n$, $1 < p_j, \theta \leq \infty$, $j = 1, \ldots, n$ the function $f(x) \in S^{r,\chi}_{p,\theta} B(R^n : E)$ be represented as*

$$f(x) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} Q_{k_1,...k_m}(x)$$

where $Q_{k_1,\dots k_m}(x)$ *are entire functions of powers of* 2^{k_j}, $j = 1,\dots,m$ *with values in* E *satisfying the inequalities*

$$\sum_{e^1+e^2=e_r} 2^{-\sum\limits_{j\in e^2} N_j} \Big\{ \sum_{0\le k_j\le N_j, j\in e^2} \cdots \sum_{N_j<k_j\le\infty, j\in e^1} \cdots 2^{\theta\left[\sum\limits_{j\in e^2} k_j(r_j+1)+\sum\limits_{j\in e^1} k_j r_j\right]} \times$$

$$\times \|Q_{k^e}\|^{\theta}_{L_p(R^n:E)}\Big\}^{\frac{1}{\theta}} \le c_1 M\chi(2^{-N}), \tag{1}$$

where c *is independent of* M, $\chi(2^{-N}) = \chi_1(2^{-N_1}),\dots,\chi_n(2^{-N_n})$.

Theorem 2. *If* $Q_{k_1,\dots,k_m}$ *are entire functions of degree* 2^{k_j}, $j = 1,2,\dots,m$ *with values from* E *whose norms satisfy conditions (1)* $1 \le p \le \infty, 1 \le \theta \le \infty$ *then the function*

$$f(x) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} Q_{k_1,\dots,k_m}(x) \tag{2}$$

where the series converges in the metric $L_p(R^n : E)$ *belongs to the class* $S^{r,\chi}_{p,\theta}B(R^n : E)$ *and the inequality*

$$\|f : S^{r,\chi}_{p,\theta}B(R^n : E)\| \le cM^r_{p,\theta}.$$

Theorem 3. *If* $\rho = (\rho_1,\dots\rho_n)$, $r = (r_1,\dots,r_n)$, $0 < \rho_j < r_j$, $1 < p_j,\theta < \infty$, $j = 1,\dots,n$, *then we have an embedding*

$$S^{r,\chi}_{p,\theta}B(R^n : E) \subset S^{\rho,\chi}_{p,\theta}B(R^n : E)$$

$$\|f; S^{\rho,\chi}_{p,\theta}B(R^n : E)\| \le c\|f; S^{r,\chi}_{p,\theta}B(R^n : E)\|$$

Theorem 4. *Let* $e_r = e_n$, $1 \le p_j,\theta \le \infty$, $1 \le p_j \le p'_j \le \infty$, $\lambda = (\lambda_1,\dots,\lambda_n)$ *for which*

$$\rho = r - \lambda - \Big(\frac{1}{p_j} - \frac{1}{p'_j}\Big)\omega_n > 0$$

Then if $f(x) \in S^{r,\chi}_{p,\theta}B(R^n : E)$, *then* $f^{\lambda}(x) \in S^{\rho,\chi}_{p',\theta}B(R^n : E)$ *and*

$$\|f; S^{\rho,\chi}_{p',\theta}B(R^n : E)\| \le c\|f; S^{r,\chi}_{p,\theta}B(R^n : E)\|$$

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Some results of multiparameter spectral theory

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The multiparameter system

$$\left(A_{i,0} + \lambda_1 A_{i,1} + \cdots + \lambda_n A_{i,n}\right) x_i = 0, \tag{1}$$

where $A_{i,k}$ are bounded operators acting in the separable Hilbert space $H_i (i = 1, 2, ..., n), \lambda = (\lambda_1, ..., \lambda_n) \in C_n, H = H_1 \otimes \cdots \otimes H_n$ is considered.

Definition 1. *Let such nonzero element $x_i \in H_i$ exists, that (1) is satisfied. Then element $x_1 \otimes \cdots \otimes x_n$ is named by eigenvector of (1) and $\lambda = (\lambda_1, ..., \lambda_n)$ is the corresponding eigenvalue.*

Definition 2. *Let be $m_1, m_2, ..., m_n$ natural numbers. Element $z_{m_1,m_2,...,m_n} \in H$ is named by $(m_1, m_2, ..., m_n)$- th associated vectors to eigenvector $z_{0,0,...,0}$ of system (1) corresponding to eigenvalue λ_0, if*

$$\left(z_{i_1,i_2,...,i_n}\right) \subset H_1 \otimes \cdots \otimes H_n, 0 \le i_k \le m_k, k = 1, 2, ..., n$$

and

$$A_k^+(\lambda_0) z_{i_1,i_2,...,i_n} + A_{k,1}^+ z_{i_1-1,i_2,...,i_n} + ... + A_{k,n}^+ z_{i_1,...,i_{n-1},i_n-1} = 0, k = 1, 2, ..., n$$

are true. A_i^+ is the operator induced to the space H by the operator A_i acting in the space H_i. Δ_i $(i = 1, 2, ..., n)-$ abstract analogue of determinant of (1) and all operators Δ_i act in space H.

Theorem. *Let $k_1, ..., k_s$ $(s \le n; k_i \le n)$ be different positive numbers and $\hat{z}$ is the common eigenvector of all operators $\Gamma_{k_i} (i = 1, ..., s)$:*

$$\Gamma_{k_i} \hat{z} = \lambda_{k_i}^0 \hat{z}.$$

Then there is an eigenvalue $(\lambda_1, .., \lambda_{k_1}^0, ..., \lambda_n)$ of the system (1) and $\hat{z}$ is the linear combination of eigenvectors and corresponding associated vectors of (1) in all directions $(m_1, m_2, ..., m_n)$ when $m_{k_1} = m_{k_2} = ... = m_{k_s} = 0$.

Multicontinuum homogenization and applications

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One of the commonly used approaches for multiscale problems includes homogenization and its variations, where effective properties at each macroscale grid or point are computed. These computations are often based on local solutions computed in a representative volume element (or coarse grid) centered at a macroscale point. Homogenization-based approaches assume scale separation and that the local media can be replaced by a homogeneous material. As a result, it is assumed that the solution average in each coarse block approximates the heterogeneous solution within that coarse block.

In many cases, even within the scale separation realm, homogenization (as discussed above) is not sufficient and the coarse-grid formulation requires multiple homogenized coefficients. These approaches are developed for different applications and we call them (following the literature) multicontinuum approaches. Multicontinuum approaches assume that the solution average is not sufficient to represent the heterogeneous solution within each coarse block. In the derivation of multicontinuum approaches, there are typically several assumptions: (1) continua definitions; (2) physical laws describing the interaction among continua; and (3) conservation laws deriving final equations. Various assumptions are typically made in deriving these models. The first such approach is presented in Rubinsteinâ€™s work (in 1948), where the author assumes existence of continua that have different equilibrium temperatures among each other (continua) and formulates empirical laws for interaction among continua.

In our earlier works, we define the continua via local spectral decompositions and show that the resulting approach converges independent of scales and contrast if representative volumes are chosen to be coarse blocks. In the current work, we use similar ideas for problems with scale separation and formulate cell problems and formally derive multicontinuum equations.

The main objectives in this talk are the following.

We derive multicontinuum methods using a homogenization-like expansion and present constraint cell problem formulations.

Constraint cell problems allow using averages for different quantities and regions (continua) and give flexibility to the framework.

We discuss appropriate local boundary conditions in representative volume elements for problems with scale separationand introduce oversampling. Using oversampling, we consider reduced constraint cell problems, where we use constraints for the averages only.

The resulting multicontinuum equations show that local averages of the solution will differ among each other if diffusion and reaction terms in the upscaled equations balance each other. This requires smaller reaction and/or larger diffusion terms, which occur in the presence of high contrast. We discuss this issue and show that a multicontinuum concept is via local spectral decomposition.

We discuss the relation to NLMC approaches that go beyond scale separation. The average constraints are easy to set and guarantee exponential decay (i.e., we remove boundary effects).

We note that to go beyond scale separation, numerical approaches use entire coarse blocks to do local computations. Among these approaches, multiscale finite element method and its variations are proposed, where multiscale basis functions are computed on coarse grids. Multiscale finite element methods construct multiscale basis functions to approximate the solution on a coarse grid. This approach has some disadvantages. To generalize this approach to more complex heterogeneities and the multi- continua case, Generalized Multiscale Finite Element Method (GMsFEM) is proposed. GMsFEM proposes a systematic approach to compute multiple basis functions. This approach starts with a space of snapshots, where one performs local spectral decomposition to compute multiscale basis functions. Adaptivity can be used to select basis functions in different regions. Each mul tiscale basis function represents a continua as discussed in [8] and there is no need for coupling terms between these continua. The basis functions for each continua are automatically identified.

The GMsFEM approach has been used jointly with localization ideas, where the authors propose Constraint Energy Minimizing GMsFEM. In this approach, oversampling regions are used to compute the multiscale basis functions. This construction takes into account spectral basis functions to localize the computations. The localization is restricted to log(H) layers and depends on the contrast, which can be reduced using snapshot functions. Moreover, it was shown that the approach converges independent of the contrast and the convergence is linear with respect to the coarse mesh size. More precisely, the convergence is proportional to H/Λ, where Λ is associated with the smallest eigenvalue that the corresponding eigenfunction is not included in the coarse space. Note that basis functions associated to fractures correspond to very small eigenvalues.

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On discreteness of the spectrum of a degenerate elliptic-differential operator

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Let $G \subset R^n$ be a bounded domain and ∂G be its boundary that we consider as a piecewise smooth surface. Let us consider the functional

$$J[u] = \frac{1}{2}[(lu,u) + (u,lu)] \ , \ \ u \in C_0^\infty (G),$$

where

$$l = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

for each $x \in G$, $A(x) = (a_{ij}(x))_{i,j=1}^n$ is a positive –definite matrix with the elements from the class $C^{(1)}(G)$, $b_i(x)$ $(i = 1, 1, ..., n)$ are real functions from the class $C^{(1)}\left(\bar{G}\right)$, the real-valued function $c(x)$ is lower bounded and belongs to the class $C(G)$.

Definition. *Let $x^{(0)} \in \partial G$. If there exists such a sequence $\left\{x_k^{(0)}\right\} \subset G$, that $\lim\limits_{k\to\infty} x_k^{(0)} = x^{(0)}$ and for some $\xi^{(0)} = \left(\xi_1^{(0)}, \xi_2^{(0)}, ..., \xi_n^{(0)}\right) \neq 0$,*

$$\lim_{k\to\infty} \sum_{i,j=1}^{n} a_{ij}\left(x_k^{(0)}\right) \xi_i^{(0)} \xi_j^{(0)} = 0,$$

the point $x^{(0)}$ is a called a singular boundary point.

In the paper, using the Lax-Milgram theorem [1], by means of the bilinear from $J[u,v]$ we construct a self-adjoint operator L. Then, in the presence of singular boundary points, using the works [2,3], we prove the discreteness of the spectrum of the operator L.

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Algebraic classification method of means

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When teaching statistics we often refer to several notions of mean, such as arithmetic, geometric, harmonic and power mean. The question appears what is actually a mean? Making use of the algebraic classification of basic physical quantities like road, speed, and acceleration we will show which mean is proper for each of the mentioned quantities.

Regularity for elliptic equations under minimal assumptions

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We give a review on regularity results for linear and quasilinear uniformly elliptic equations. Focus is on the *minimal assumptions* we need to obtain a given degree of smoothness for generalized solutions of a given elliptic equation. Our main linear equation is

$$A(x, u, \nabla u) + B(x, u, \nabla u) = 0$$

where

$$\begin{cases} |A(x,u,\xi)| \leq a|\xi|^{p-1} + b|u|^{p-1} + e \\ |B(x,u,\xi)| \leq c|\xi|^{p-1} + d|u|^{p-1} + g \\ \xi \cdot A(x,u,\xi) \geq |\xi|^p - d|u|^p - g \end{cases}$$

where a is a real constant and b, c, d, e, g are given functions.

Noncyclic vectors of coanalytic Toeplitz operators and related questions

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Abstract. We describe noncyclic vectors of coanalytic Toeplitz operator T on the Hardy space $H2(D)$ in terms of commutant of the operator T. Namely, we prove that the function f in $H2(D)$ is a noncyclic vector for the operator T if and only if there exists a nonzero operator A in the commutant of T of the operator T such that $Af = 0$. This gives a concrete answer to Nikolskii's question posed in his book. Some other related questions are also discussed.

Studying some effects arising in inhomogeneous media during the passage of non-stationary waves

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Problem statement. We consider a periodically inhomogeneous plane –laminar medium and derive an equation that describes the process of wave propagation in the direction Oz, perpendicular to the plane of layers. In such a statement, the stress-strain state will be uniaxial with the Oz axis, the equation of motion will be of the form [1, p. 35, 2, p. 17]:

$$\frac{\partial \sigma}{\partial z} = \rho(z) \frac{\partial^2 \vartheta}{\partial t^2}, \tag{1}$$

where

$$\sigma(z,t) = E(z) \cdot \frac{\partial \vartheta(z,t)}{\partial z}, \tag{2}$$

$\sigma(z,t)$ is stress, $\vartheta = \vartheta(\vartheta, t)$ is displacement, $\rho(z)$ is the density of the soil arrey material, $E(z)$ is Young's modules.

Proceeding from the physical characteristics of the soil array, we will assume

$$\begin{array}{ll} E(z) = E_0 + \nu_0 \cos(T_0 z), & E_0 = const \\ \rho(z) = \rho_0 + \nu_1 \cos(T_0 z), & \rho_0 = const \end{array} \tag{3}$$

where ν_0, ν_1 are small (compared to E_0, ρ_0), $T_0 = \frac{2\pi}{k}$, k is in homogeneity period along the axis Oz. Introducing the denotations $c_0^2 = E_0/\rho_0$; $\epsilon = \frac{\nu_1}{\rho_0} - \frac{\nu_0}{E_0}$, we will have $\epsilon << 1$. After substituting (2) and (3) in (1), assuming that the quantity $\frac{\partial E}{\partial z} \frac{\partial \vartheta}{\partial z}$ is negligibly small compared to the quantities $E_0 \frac{\partial^2 \vartheta}{\partial z^2}$, $\nu_0 \frac{\partial^2 \vartheta}{\partial z^2}$, $\rho \frac{\partial^2 \vartheta}{\partial t^2}$ $\nu_1 \frac{\partial^2 \vartheta}{\partial t^2}$, ignoring the quantities of order ϵ^2 and leaving the quantities of order ϵ, we obtain an equation of motion of medium in displacements :

$$\frac{\partial^2 \vartheta}{\partial t^2} = \frac{1}{c_0^2} (1 + \varepsilon \cos(T_0 z)) \cdot \frac{\partial^2 \vartheta}{\partial z^2}. \tag{4}$$

In the case of harmonic seismic wave, we will look for the solution in the form:

$$\vartheta(z,t) = A(z) \cdot e^{i\omega t}. \tag{5}$$

where ω is the frequency of the seismic wave. Allowing for (5) in (4) we obtain:

$$A''(z) + \frac{\omega^2}{c_0^2}\left(1 + \varepsilon \cos\left(T_0 z\right)\right) A(z) = 0 \tag{6}$$

Problem solution. In principle, for solving this equation, we can use the small parameter method but the use of theory of Mathieu and Hill equations became more effective for revealing mechanical effects.

Equation (6) takes the standard form of the Mathieu equation [2, p. 604]. Introducing the denotations

$$\frac{T_0}{z} \equiv \xi; \quad \frac{\omega^2}{c_0^2} = \eta; \quad \eta\varepsilon = \gamma \tag{7}$$

instead of (6) we have:

$$\frac{d^2 A}{d\xi^2} + \left(\eta + \gamma \cos\left(2\xi\right)\right) A(\xi) = 0 \tag{8}$$

By the Floke theorem [2, p. 234] the general solution of the equation (8) is of the form:

$$A = c_1 F_1(\xi)\, e^{n\xi} + c_2 \cdot F_2(\xi) \cdot e^{-n\xi} \tag{9}$$

where c_1, c_2 are arbitrary constants, F_1, F_2 are periodic functions of the variable ξ with period $\pi; n = const$, is determined by η and γ.

The solutions of the Mathieu equation are the Mathieu functions [3, p. 320], that were studied well and tabulated. To explain the effects arising when taking into account the inhomogeneity of the medium, the whole plane of variables (η, γ) is spilled into three subdomains. In domain I, where $\eta < -\gamma$ propagating waves can not exist, in domains II and III, in the shaded waves, zones n is a purely imaginary quantity, consequently, in this case, the solution (9) will be a superposition of two undamped waves propagating in opposite directions, in the shaded zones of subregions of two undamped waves propagating in opposite directions, in shaded zones of subdomains II and III the quantity p is complex, consequently here the propagating waves damp exponentially these zones are called non-transparency zones. Thus, the medium with periodic inhomogeneity works as a filter, passing the waves with one frequency and damping the waves with frequencies corresponding to non-transparency zones. The dependence of the number n and ω in the general case is non-linear, consequently, the propagating waves disperse.

For a piecewise homogeneous medium, we obtain

$$A(z) = \begin{cases} c_1 e^{ik_1 z} + D_1 e^{-ik_1 z}, & -l_1 < x < 0 \\ c_2 e^{ik_2 z} + D_2 e^{-ik2z}, & 0 < x < l_2 \end{cases} \tag{10}$$

where $k_1^2 = k^2\left(1+\varepsilon f_1\right)$, $,k_2^2 = k^2\left(1+\varepsilon f_2\right)$, $k = \omega^2/c_0^2$ is a wave number.

By the Floke theorem for the Hill equation for a wave propagating in the positive direction of the axis Oz, there should be

$$A\left(z\right) = F\left(z\right)\, e^{i\bar{k}z} \tag{11}$$

where $\bar{k}$ is a desired wave number, $F\left(z\right)$ is periodic with a period $d = l_1 + l_2$. Comparing (11) and (12) in the interval $l_2 < z < d$ and using the equality $F\left(z-d\right) = F\left(z\right)$, we obtain the expression

$$A\left(z\right) = c_1 e^{i\bar{k}d}\cdot e^{ik_1(z-d)} + D_1 e^{i\bar{k}d}\cdot e^{-ik_1(z-d)}. \tag{12}$$

Requiring continuity of the a “amplitude” $A\left(z\right)$ and its first derivative on the interface of layers with various properties, we obtain the equation:

$$2e^{i\bar{k}d}\left(\cos\left(k_1l_1\right)\cdot\cos\left(k_2l_2\right) - \frac{1}{2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\cdot\sin\left(k_1l_1\right)\cdot\sin\left(k_2l_2\right)\right) = e^{2i\bar{k}d} - 1. \tag{13}$$

The roots $z_1 = e^{i\bar{k}_1d}$ and $z_2 = e^{i\bar{k}_2d}$are conjugated, consequently, $z_1\cdot z_2 = 1$, whence with regard to the properties of the roots of the square equations, we find

$$z_1 + z_2 =$$

$$2\left(\cos\bar{k}d\right) = 2\left(\cos\left(k_1l_1\right)\cdot\cos\left(k_2l_2\right) - \frac{1}{2}\left(\frac{k_1}{k_2}+\frac{k_2}{k_1}\right)\cdot\sin\left(k_1l_1\right)\cdot\sin\left(k_2l_2\right)\right). \tag{14}$$

(the index $y\bar{k}$ is omitted).

From the last relation, we can find non-transparency zones (blocking or closing) determining the frequency range at which the expression in the parenthesis in (14) will be greater than a unit by modules. Thus, piecewise-homogeneous media [4, p.117] also possess the property to filter wawes in certain strips. For real soil arrays the transparency zones appear at frequencies about $10^4 \div 10^8$Hertz .

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Continuity of generalized Riesz potentials

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Let X be a set together a quasi – metric ρ and a nonnegative Borel measure μ on X with supp $\mu = X, diamX = \infty$, there exists a constant $C \geq 1$ such that $C^{-1}r^N \leq \mu(B(x,r)) \leq Cr^N$ and f be a μ - locally integrable function on X. Define the operator

$$R_K(x) = \int_X K(\rho(x,y))f(y)d\mu(y) \tag{1}$$

Where $K : (0,\infty) \to [0,\infty)$ be a continuous function satisfying the conditions
1) $K(t)$ is an almost decreasing function.
2) there exists a constant $M \geq 1$ such that $K(r) \leq MK(2r)$
3) there exists a constant C and $0 < \sigma < 1$ such that

$$\int_{B(x,\ r)} K(\rho(x,y))\mu(y) < Cr^\sigma, \text{ for any } r > 0$$

It is known that the integral (1) is finite μ - almost everywhere on X, if

$$\int_X K(\rho(1 + x_0, y))f(y)d\mu(y) < \infty \tag{2}$$

Where x_0 is any fixed point on X. (2)

Theorem. *Let condition (2) be satisfied. Then the operator (1) is continuous on X.*

A mixed problem for parabolic equations with total conditions

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In this paper, we study a mixed problem for parabolic systems with discontinuous coefficients of the first kind with total conditions.

For the sake of simplicity of notation and reasonings, we consider the following model problem.

Find the solution $u = u(x,t)$ to the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < 1, 0 < t \leq T, \tag{1}$$

Satisfying the boundary conditions

$$u|_{x=0} = 0, \, u|_{x=1} = 0, \tag{2}$$

and the total condition

$$\int_0^T u(x,t)dt = \omega(x), 0 < x < 1, \tag{3}$$

where $T(T > 0)$,$a^2(Rea^2 > 0)$ are some constants, $\omega(x)$ is the given function.

Solution. Let problem (1)-(3) have a classical solution. Then integrating (1), we have

$$\int_0^T \frac{\partial u(x,t)}{\partial t} dt = a^2 \int_0^T \frac{\partial^2 u(x,t)}{\partial x^2} dt.$$

Hence, we get

$$a^2 \frac{\partial^2}{\partial x^2} \int_0^T u(x,t)dt = u(x,T) - u(x,0)$$

therefore, using (3), we have

$$f^{(T)}(x) - f^{(0)}(x) = a^2 \frac{d^2 \omega(x)}{dx^2}, \tag{4}$$

where

$$f^{(0)}(x) \equiv u(x,0), \tag{5}$$

$$f^{(T)}(x) \equiv u(x,T), \tag{6}$$

which are to be determined.

Expanding the functions , $f_k^{(0)}$, $f_k^{(T)}$ and $\omega''(x)$ in eigenfunctions of problem (1)-(2), we obtain [1]

$$f^{(0)}(x)=\sum_{k=1}^{\infty} f_k^{(0)}\sin k\pi x,\ \ f^{(T)}(x)=\sum_{k=1}^{\infty} f_k^{(T)}\sin k\pi x,\ \ \omega^{''}(x)=\sum_{k=1}^{\infty} g_k\sin k\pi x, \qquad (7)$$

where $g_k=2\int_0^1\omega''(x)\sin k\pi x$,

$f_k^{(0)}$,$f_k^{(T)}$ are unknown coefficients. According to [1], under certain conditions, the mixed problem (1),(2),(5) (if $f^{(0)}(x)$ is known) has a unique solution and it is represented by the formula

$$u(x,t)=\sum_{k=1}^{\infty} f_k^{(0)}\exp(-k^2\pi^2a^2T)\sin k\pi x,\ 0<x<1,\ 0<t\le T. \qquad (8)$$

For $t=T$ from (8) we obtain

$$f^{(T)}(x)=\sum_{k=1}^{\infty} f_k^{(0)}\exp(-k^2\pi^2a^2T)\sin k\pi x, \qquad (9)$$

Taking into account (7) in (4) and (9), to determine the coefficients $f_k^{(0)}$ and $f_k^{(T)}$ we obtain the systems of equations

$$\left.\begin{array}{l} f_k^{(T)}-f_k^{(0)}=a^2g_k \\ f_k^{(T)}=\exp(-k^2\pi^2a^2T)f_k^{(0)} \end{array}\right\} \qquad (10)$$

Solving systems (10) we obtain

$$f_k^{(0)}=-\frac{a^2g_k}{1-\exp(-k^2\pi^2a^2T)}, \qquad (11)$$

Thus, we have established the following

Theorem 1. *Let $\omega(x)\in C^2([0,1])$ and $\omega''(x)$ be a function piece-wise absolutely continuous in the segment $[0,1]$ and $\frac{d^3\omega(x)}{dx^3}\in L_p(0,1),p>1$. Then, for $Rea^2>0$ if the problem (1)-(3) has a classical solution, then it is unique and this solution is represented by formula (8), where $f_k^{(0)}$ is from (11). The following theorem can be easily proved by direct verification.*

Theorem 2. *Let $\omega(x)\in C^4([0,1])$ and the matching conditions*

$$\omega''(x)|_{x=0}=\omega''(x)|_{x=1}=0.$$

Then for $Rea^2>0,(a^2-const)$, be satisfied problem (1)-(3) has a unique classical solution; this solution is represented by formula (8), where $f_k^{(0)}$ is from (11).

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Method of generalization of trigonometric functions

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One of the angels of the triangles considered in the paper equals the constant number ω here the number ω is an arbitrary fixed number satisfying the inequality $\frac{\pi}{2} \leq \omega < \pi$.

Let us consider the triangles ABC and $A_1B_1C_1$ one of whose angles is the number α ($0 < \alpha < \pi - \omega$) (fig.1.)

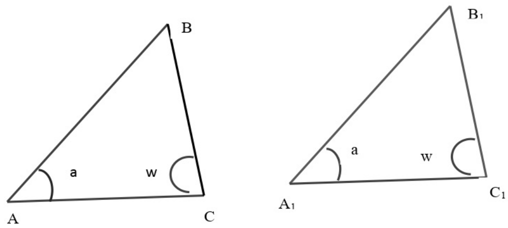

Fig 1.

As the sum of inner angles of the triangle equals the constant number π, $\angle B = \angle B_1$. So, these triangles are similar:

$$\angle ABC \sim \angle A_1B_1C_1 \tag{1}$$

If follows from (1) that [1], [2]

$$\frac{AB}{A_1B_1} = \frac{AC}{A_1C_1} = \frac{BC}{B_1C_1} \tag{2}$$

From (2), we obtain

$$\frac{BC}{AB} = \frac{B_1C_1}{A_1B_1}, \tag{3}$$

Thus, the values of the fraction $\frac{BC}{AB}$ depend only on the size of the degree of α, but not on the size of the triangle. Since the ratio $\frac{BC}{AB}$ is the same number for the fixed angle α we can denote it by the same symbol.

Definition 1. We call the ratio of the side (BC) facing the angle α in the triangle ABC (fig. 1) to the side (AB) facing the angle ω the E- sine of the angle ω and write it as sinus $E\sin\alpha \ = \frac{BC}{AB}$.

From the equalities (2) we obtain the equalities

$$\frac{AC}{AB} = \frac{A_1C_1}{A_1B_1}\ , \quad \frac{BC}{AC} = \frac{B_1C_1}{A_1C_1}\ , \quad \frac{AC}{BC} = \frac{A_1C_1}{B_1C_1} \tag{4}$$

These equalities show that the numerical value of the ratios contained in (4) depends not on the lengths of the sides of the triangle, but only on the size of the degree of the angle α .

We call the side facing the angles ω in the triangle ABC the E-hypotenuse, the other sides the E-catheter.

Definition 2. We call the ratio of the E-catheter (AC) adjacent to the angle α to the E-hypotenise (AB) the E-cone of the angle α and write it as $Ecos\alpha = \frac{AC}{AB}$.

Definition 3. We call the ratio of the E-catheter (BC) facing the angle α in the triangle ABC to the adjacent E-catheter (AC) the E-tangent of the angle α and write it as $Etg\alpha = \frac{BC}{AC}$.

Definition 4. We call the ratio of the E-catheter (AC) adjacent to the angle α in the triangle ABC to the facing E catheter (BC) the E-cotangent of the angle α and write it as $Ectg\alpha = \frac{AC}{BC}$.

Remark. Since $\frac{\pi}{2} \leq \omega < \pi$ then $0 < \alpha < \omega$. Hence, $BC < AB$ and $AC < AB$. So, for arbitrary α $(0 < \alpha < \pi - \omega)$ the inequalities

$$0 < Esin\alpha < 1, 0 < Ecos\alpha < 1$$

are satisfied.

From the above statements, we obtain the validity of the following identities.

Theorem 1. *For* $0 < \alpha < \pi - \omega$ *, the identities* $Etg\alpha = \frac{Esin\alpha}{Ecos\alpha}$, $Ectg\alpha = \frac{Ecos\alpha}{Esin\alpha}$ *are valid.*

Theorem 2. *For* $0 < \alpha < \pi - \omega$ *the indentity*

$$Esin^2\alpha + Ecos^2\alpha - 2Esin\alpha \cdot Ecos\alpha \cdot Ecos\omega = 1 \tag{5}$$

is valid.

Proof: According to the theorem of cosines, in the triangle ABC we can write

$$AB^2 = BC^2 + AC^2 - 2AB \cdot AC \cdot cos\omega$$

Hence we obtain the identity:

$$\left(\frac{BC}{AB}\right)^2 + \left(\frac{AC}{AB}\right)^2 - 2\frac{BC}{AB} \cdot \frac{AC}{AB} \cdot cos\omega = 1 \ , \tag{6}$$

Writing the equalities $Esin\alpha = \frac{BC}{AB}$, $Ecos\alpha = \frac{AC}{AB}$ in (6) we get the validity of the identity (5).

We prove the following theorem.

Theorem 3. For $0 < \alpha < \pi - \omega$, the identities $Esin\alpha = \frac{sin\alpha}{sin\omega}$, $Ecos\alpha = cos\alpha + ctg\omega \cdot sin\alpha$ are valid.

There arises such a question: can we define the functions $y = Esinx$ and $y = Ecosx$ on the whole number axis so that the properties of these functions can correspond to the properties of the functions $y = sinx$ and $y = cosx$ in some sense?

Theorem 4. If the number $\frac{\pi}{2(\pi-\omega)} = k$ is a natural number, then the functions $y = Esinx$ and $y = Ecosx$ can be determined on the whole number axis so that the properties of these functions correspond to the properties of the functions $y = sinx$ and $y = cosx$.

Theorem 5. For $\frac{\pi}{2} \leq \omega \leq \pi$ and $\frac{\pi}{2(\pi-\omega)} = k$ being a natural number, the functions y=Esinx and y=Ecosx are continuous functions determined on the real axis and

$1^0. -1 \leq Esinx \leq 1, x \in (-\infty; \infty)$

$2^0. -1 \leq Ecosx \leq 1, x \in (-\infty; \infty)$

3^0. $y = Esinx$ and $y = Ecosx$ are periodic functions and the least positive period is $T = 4(\pi - \omega)r$.

$$Esin(x + m \cdot 4\,(\pi - \omega)) = Esinx,$$

$$Ecos(x + m \cdot 4\,(\pi - \omega)) = Ecosx,\ x \in (-\infty; \infty)\,,\ m = 0 \pm 1, \pm 2, \ldots;$$

4^0. $(Esinx)^2 + (Ecosx)^2 - 2\,|Esinx| \cdot |Ecosx| \cdot cos\omega = 1, x \in (-\infty; \infty)$

5^0. The function $y = Esinx$ is an odd function $Esin(-x) = -Esinx$, $x \in (-\infty; \infty)$; 6^0. The function $y = Ecosx$ is an even function $Ecos(-x) = Ecosx$, $x \in (-\infty; \infty)$;

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On the strong solvability of a nonlocal boundary value problem for the Laplace equation in a rectangular domain

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In this paper, we consider the following nonlocal boundary value problem for the Laplace equation:

$$u_{xx} + u_{yy} = 0\,, \quad 0 < x < 2\pi\,, \;\; 0 < y < h\,, \tag{1}$$

$$u\left(x, 0\right) = \varphi\left(x\right)\,, \;\; u\left(x, h\right) = \psi\left(x\right), \quad 0 < x < 2\pi, \tag{2}$$

$$u_x\left(0, y\right) = 0\,, \;\; u\left(0, y\right) = u\left(1, y\right)\,, \quad 0 < y < h\,. \tag{3}$$

in a weighted Sobolev space with a weight from the Mackenhoupt class. The notion of a strong solution of this problem is defined. The correct solvability of this problem is proved by the Fourier method.

In the work of N. I. Ionkin and E. I. Moiseev [1], for multidimensional parabolic equations, a boundary value problem was solved with nonlocal conditions supported by the characteristic and improper parts of the domain boundary. In works [2, 3], problem (1)-(3) is considered in an infinite strip in the classical formulation.

To formulate the main results, we present the definitions of some weighted spaces. Let $\nu : [0,\ 2\pi] \to (0, +\infty) -$ be some weight function, $\Pi = (0,\ 2\pi) \times (0, h)$. Denote by $L_{p;\nu}\left(\Pi\right)$ the Banach space of functions on $\Pi$with mixed norm

$$\|f\|_{L_{p,\nu}(\Pi)} = \int_0^h \left(\int_0^{2\pi} |f\left(x;y\right)|^p \nu\left(x\right) dx \right)^{\frac{1}{p}} dy\ , \quad 1 < p < +\infty.$$

We denote by $W_{p,\nu}^2\left(\Pi\right)$ the Sobolev space generated by the norm

$$\|u\|_{W_{p;\nu}^2} = \sum_{|\alpha| \le 2} \|\partial^\alpha u\|_{L_{p,\nu}(\Pi)}\,.$$

Weighted Lebesgue space generated by the norm

$$\|f\|_{L_{p,\nu}(I)} = \left(\int_I |f\left(x\right)|^p \nu\left(x\right) dx \right)^{\frac{1}{p}}.$$

denote by $L_{p,\nu}\left(I\right)$, where $I = (0,\ 2\pi)$. We also consider the weighted Sobolev space $W_{p,\nu}^2\left(I\right)$, generated by the norm

$$\|f\|_{W_{p,\nu}^2(I)} = \|f\|_{L_{p,\nu}(I)} + \|f'\|_{L_{p,\nu}(I)} + \|f''\|_{L_{p,\nu}(I)}\,.$$

Recall that the class of Mackenhoupt weights $A_p(I)$ is the class of periodic functions (i.e., it is considered that the function $\nu(x)$ is periodically extended to the entire axis with period 2π) satisfying the condition

$$\sup_{J\subset I}\left(\frac{1}{|J|}\int_J \nu(t)\,dt\right)\left(\frac{1}{|J|}\int_I |\nu(t)|^{-\frac{1}{p-1}}\right)^{p-1}<+\infty,$$

where the supremum is taken over all intervals $J\subset I$and $|J|-$ is a length of the interval J.

Definition. *A functionu $\in W^2_{p,\nu}(\Pi)$ is called a strong solution to problem (1)-(3) if equality (1) is satisfied a.e. $(x;y)\in\Pi$, and its trace $u|_{\partial\Pi}$ satisfies relations (2), (3).*

Let us introduce into consideration the system of functions$\{u_n(x)\}_{n\in Z^+}$ and $\{\vartheta_n(x)\}_{n\in Z^+}$, where

$$u_{2n}(x)=\cos nx,\ n\in Z^+,\quad u_{2n-1}(x)=x\sin nx,\ n\in N, \tag{4}$$

$$\vartheta_0(x)=\frac{1}{2\pi^2}(2\pi-x),\vartheta_{2n}(x)=\frac{1}{\pi^2}(2\pi-x)\cos nx,\vartheta_{2n-1}(x)=\frac{1}{\pi^2}\sin nx\,,\ n\in N. \tag{5}$$

Note that systems (4) and (5) are biorthogonally conjugate, which can be verified directly. In obtaining the main result, the following theorem is essentially used.

Theorem 1. *Let $\nu\in A_p(I)$, $1<p<+\infty$. Then system (4) forms a basis in $L_{p,\nu}(I)$.*

The solution to problem (1)-(3) is sought in the form of a series

$$u(x,y)=U_0(y)+\sum_{n=1}^{\infty}U_n(y)\,u_n(x)\,,$$

where

$$U_0(y)=\frac{\psi_0-\varphi_0}{h}y+\varphi_0,U_{2n-1}(y)=\psi_{2n-1}\frac{\sinh ny}{\sinh nh}+\varphi_{2n-1}\frac{\sinh n(h-y)}{\sinh nh},$$

$$U_{2n}(y)=(\psi_{2n}-(h-y)\psi_{2n-1})\frac{\sinh ny}{\sinh nh}+(\varphi_{2n}+y\varphi_{2n-1})\frac{\sinh n(h-y)}{\sinh nh},\ n\in N,$$

$$\varphi_n=\int_0^{2\pi}\varphi(x)\,\vartheta_n(x)\,dx,\quad \psi_n=\int_0^{2\pi}\psi(x)\,\vartheta_n(x)\,dx,\quad n\in Z^+.$$

The main result of the paper is the following theorem

Theorem 2. *Let $\nu\in A_p(I)$, $1<p<+\infty$, the boundary functions $\varphi(x)$ and $\psi(x)$ belong to the space $W^2_{p,\nu}(I)$ and satisfy the conditions*

$$\varphi(2\pi)-\varphi(0)=\varphi'(0)=0,\ \psi(2\pi)-\psi(0)=\psi'(0)=0.$$

Then problem (1)-(3) has a unique solution in $W^2_{p,\nu}(\Pi)$ and moreover it is valid the following estimate

$$\|u\|_{W^2_{p;\nu}(\Pi)}\le c\left(\|\varphi\|_{W^2_{p;\nu}(I)}+\|\psi\|_{W^2_{p;\nu}(I)}\right),$$

where $c>0$ is a constant independent of $\varphi(x)$ and $\psi(x)$.

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On absolutely and uniformly convergence of the spectral expansion for one boundary value problem of a second-order differential operator

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Consider the following boundary value problem

$$ly = -y'' + q(x)\, y = \lambda y, x \in G = (0,1) \tag{1}$$

$$\alpha y(0) + \beta y'(0) - y(1) = 0 \tag{2}$$

$$\gamma y(0) + \delta y'(0) - y'(1) = 0 \tag{3}$$

where is $q(x) \in L_1(G) -$ a real function, $\alpha = c_1 e^{i\theta}$, $\beta = c_2 e^{i\theta}$, $\gamma = c_3 e^{i\theta}$, $\delta = c_4 e^{i\theta}$; real numbers, $i = \sqrt{-1}$. Self-adjoint conditions has the form $c_1 c_4 - c_2 c_3 = 1$.

Let $\{y_k(x)\}_{k=1}^{\infty}$ – be a complete orthonormal system in $L_2(G)$ consisting of eigenfunctions and $\{\lambda_k\}_{k=1}^{\infty}$ – be the corresponding system of eigenvalues of boundary value problem (1)-(3).

By $W_1^1(G)$, we denote the class of functions $f(x)$, absolutely continuous on the interval $\overline{G}$ for which $f'(x) \in L_p(G)$. Denote $\mu_k = \sqrt{\lambda_k}$, introduce a partial sum of the orthogonal expansion of the function $f(x) \in W_1^1(G)$, with respect to the system $\{y_k(x)\}_{k=1}^{\infty}$:

$\sigma_\nu(x,f) = \sum\limits_{\mu_k \le \nu} f_k\, u_k(x)$, $\nu > 0$, where $f_k = (f, u_k) = \int\limits_0^1 f(x)\, \overline{u_k(x)} dx$.

We introduce the difference $R_\nu(x,f) = f(x) - \sigma_\nu(x,f)$.

Teorem 1. *Assume that $\beta \ne 0$, a function $f(x)$ belongs to the class $W_1^1(G)$, and the inequality $\sum\limits_{n=2}^{\infty} n^{-1}\omega_1\left(f', n^{-1}\right) < \infty$ are satisfied. Then the spectral expansion of the function with respect to the system $\{y_k(x)\}_{k=1}^{\infty}$ converges absolutely and uniformly on the interval $\overline{G} = [0,1]$ and the estimate*

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \le const \left\{ C_1(f)\, \nu^{-1} + \sum_{n=[\nu]}^{\infty} n^{-1}\omega_1\left(f', n^{-1}\right) + \right.$$
$$\left. + \omega_1\left(f', \nu^{-1}\right) + \nu^{-1}\left(\|f\|_\infty + \left\|f'\right\|_1\right)\|q\|_1 + \left\|f'\right\|_1 \right\} \tag{4}$$

holds, where $\nu \ge 8\pi$, $C_1(f) = \left[|\gamma| + |\alpha|\, |\delta|\, \left|\beta^{-1}\right|\right] |f(1)| + \left[\left|\beta^{-1}\right| + |\alpha|\, \left|\beta^{-1}\right|\right]\ |f(0)|$, $\omega_1\left(\cdot, n^{-1}\right)$ is the modulus of continuty of the function $f'(x)$ in the space $L_1(G)$, the constant const is independent of the function $f(x)$ and ν.

Theorem 2. *Assume that $\beta = 0$, a function $f(x)$ belongs the class $W_1^1(G)$, $f(1) - \alpha f(0) = 0$ and the inequality $\sum\limits_{n=2}^{\infty} n^{-1}\omega_1\left(f', n^{-1}\right) < \infty$ are satisfied. Then the spectral* *expansion of the function $f(x)$ with respect to the system $\{y_k(x)\}_{k=1}^{\infty}$ converges absolutely and uniformly on the interval $\overline{G} = [0, 1]$ and the estimate*

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \leq const\left\{C_2(f)\,\nu^{-1} + \sum_{n=[\nu]}^{\infty} n^{-1}\omega_1\left(f' n^{-1}\right) + \right.$$

$$\left. +\omega_1\left(f', \nu^{-1}\right) + \nu^{-1}\left(\|f\|_\infty + \|f'\|_1\right)\left(1 + \|q\|_1\right)\right\}$$

holds, where $\nu \geq 8\pi$, $C_2(f) = |\gamma f(1)|$, the constant const is independent of the function $f(x)$ and ν.

Corollary 1. If the function $f(x) \in W_1^1(G)$ in Theorems 1 and 2 satisfies the relation $f(0) = f(1) = 0$. Then the spectral expansion of the function $f(x)$ with respect to the system $\{y_k(x)\}_{k=1}^{\infty}$ converges absolutely and uniformly on the interval $\overline{G} = [0, 1]$ and the following estimate holds:

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \leq$$

$$\leq const\left\{\sum_{n=[\nu]}^{\infty} n^{-1}\omega_1\left(f', n^{-1}\right) + \omega_1\left(f', \nu^{-1}\right) + \nu^{-1}\|f'\|_1\left(1 + \|q\|_1\right)\right\}.$$

Corollary 2. If the function $f(x) \in W_1^1(G)$ in Theorems 1 and 2 satisfies the relations $f(0) = f(1) = 0$ and $f'(x) \in H_1^\gamma\,(0 < \gamma \leq 1)$, ($H_1^\gamma$ -is , the, Nikolskii class). Then the spectral expansion of the function $f(x)$ with respect to the system $\{y_k(x)\}_{k=1}^{\infty}$ converges absolutely and uniformly on the interval $\overline{G} = [0, 1]$ and the following estimate holds:

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \leq const\nu^{-\gamma}\,\|f\|_1^\gamma, \quad \nu \geq 8\pi,$$

where $\|f'\|_1^\gamma = \|f'\|_1 + \sup\limits_{\delta>0}\frac{\omega_1(f',\delta)}{\delta}$, the constant const is independent of the function $f(x)$.

Note that, the case $f(x) \in W_p^1(G)$, $p > 1$, it was considered in the work [3] author.

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Teichmüller's theorem on separating rings in higher dimensions

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By the Uniformization Theorem, a ring domain $\mathcal{R}$ (a doubly connected domain) of the complex plane $\mathbb{C}$ is conformally equivalent to the annulus $\{z \in \mathbb{C} : r_0 < |z| < r_1\}$ for some $0 \leq r_0 < r_1 \leq \infty$. The quantity $\log(r_1/r_0)$ is called the modulus of $\mathcal{R}$ and denoted by $\operatorname{mod} \mathcal{R}$.

O. Teichmüller [1] showed that a ring $\mathcal{R}$ with $\operatorname{mod} \mathcal{R} > \pi$ separating 0 and ∞ contains a circle centered at 0 and that the constant π is sharp. Indeed, the Teichmüller ring $R_T(t) = \mathbb{C} \setminus \big([-1, 0] \cup [t, +\infty)\big)$ with $t = 1$ serves as an extremal case.

Teichmüller introduced the Grötzsch ring and the Teichmüller ring and found their extremal properties in [1]. Using the extremal property of the Teichmüller ring, D. A. Herron, X. Liu and D. Minda [2] showed the following sharp result.

Theorem. *Let $\mathcal{R}$ be a ring separating* 0 *and* ∞ *in* $\mathbb{C}$ *with* $m = \operatorname{mod} \mathcal{R} > \pi$. *Then* $\mathcal{R}$ *contains an annular subring* $\mathcal{A}$ *of the form* $\{z : r_0 < |z| < r_1\}$ *with*

$$\operatorname{mod} \mathcal{A} = \log \mu_T^{-1}(m),$$

where $\mu_T(t) = \operatorname{mod} R_T(t)$ *for* $0 < t < +\infty$.
The result is sharp.

From the inequality $\mu_T(t) < \log t + \pi$ for $t > 1$, which is equivalent to $m < \log \mu_T^{-1}(m) + \pi$ for $m = \mu_T(t) > \pi$, F. G. Avkhadiev and K.-J. Wirths [3] deduced a sharp explicit form of the above theorem.

Our main goal in the present talk is to extend the Teichmüller theorem to higher dimensions (a main problem here is that there is no analogue of the Uniformization Theorem in $\mathbb{R}^n$, $n \geq 3$). In addition, we apply this result to studying boundary correspondence problems. We emphasize that our approach may allow us to weaken regularity or quasiconformal assumptions of the mappings. Such applications to mappings of finite directional dilatations will be also presented.

The talk is based on [4] and on our forthcoming paper.

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A new proof of Wiener's criterion for the heat equation

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In a bounded domain $D \subset \mathbb{R}^{n+1}$ we consider the Dirichlet problem for the heat equation

$$\triangle u - u_t = 0, \quad u|_{\partial_p D} = f \in C(\partial D) \tag{1}$$

and we get the Wiener type criterion in potential terms (also as a result in capacity terms) for the regularity of the boundary point $(x_0, t_0) \in \partial D$ with respect to this problem.

Definition. A point $(x_0, t_0) \in \partial D$ is said to be regular if

$$\lim_{D \ni (x,t) \to (x_0,t_0)} u_f\left(x, t\right) = f\left(x_0, t_0\right)$$

for any $f \in C\left(\partial D\right)$, where $u_f\left(x, t\right)$ is the generalized solution in the Wiener sense for the problem (1).

It should be noted that the Wiener-type regularity criterion in terms of the heat capacity of the boundary point for the heat equation was proved by Evans and Garipi [1]. Their method is based on the mean value theorem and can cover only the class of parabolic equations with divergent structures [2]. Our method can also cover the class of parabolic equations with a non-divergent structure.

Let's introduce some necessary concepts and notations. For simplicity of notation, we will henceforth assume $(x_0, t_0) = (\mathbf{0}, 0) \in \partial D$.

Let $B \subset R^{n+1}$, $(x, t) = (\ x_1, \ldots, x_n, t)$ and

$$P_B\left(x, t\right) = \int_B K(t - \tau, x - \xi) d\mu(\tau, \xi)$$

be heat potential with generated of Weierstrass kernel

$$K\left(t, x\right) = \left\{ \begin{array}{l} (4\pi t)^{-\frac{n}{2}} \\ 0, \ t \le 0 \end{array} \cdot exp\left\{ -\frac{|x|^2}{4t} \right\}, \ t > 0, \right.$$

here B is a Borel set and μ is a Borel measure.

Denote for $\lambda > 1$ and $m \in N \cup \{0\}$ paraboloids:

$$P_m = \left\{ (t, x) \mid |x|^2 < -\lambda^m \cdot t, t < 0 \right\}.$$

Let $B_{m,k} = (P_{m+1} \backslash P_m) \cap \left[-t_k; -\frac{3t_k}{4}\right]$, where $t_{k+1} = \frac{t_k}{4}$, $k \in N \cup \{0\}$, $t_0 > 0$, $P_0 = \emptyset$, and denote cylinders

$C_{m,k} = \{(t,x)\,|-t_k < t < 0,\ |x| < a\cdot\rho_{m,k}\}$, where $a > 0$ the chosen absolute constant depending only on the dimension n of the space and denote by $S_{m,k}$ the lateral surface of the cylinder $C_{m,k}$.

The measure μ on B called admissible ,if

$$P_B\,(t,x) = \int_B K\,(t-\tau, x-\xi)\, d\mu(\tau,\xi) \leq 1$$

in R^{n+1}.The number $cap\,(B) = sup\mu B$, where the supremum is taken by all possible admissible measures μ is called the thermal capacity of the set B.

Let's call $T_{m,k} = C_{m,k}\backslash P_m$ trapezoids and denote by $T^{(j)}_{m,k}, j = 1,2,\ldots,n_0(n)$ corresponding minimal finite partition $T_{m,k}$, for which the following

$$|x-\xi| \leq |\xi|$$

inequality is fulfilled at $(x,t) \in T^{(j)}_{m,k+1}$ and $(\xi,\tau) \in T^{(j)}_{m,k}$ for every fixed $j \in \{1,\ldots,n_0\}$.

Lemma 1. *If* $(x,t) \in T^{(j)}_{m,k+1}$ *and* $(\xi,\tau) \in T^{(j)}_{m,k}$ *, then*

$$\frac{|x-\xi|^2}{t-\tau} \leq \frac{|\xi|^2}{-\tau}. \tag{2}$$

Lemma 2. *There exist the following absolute constants* $C_1 > 0$ *and* $C_2 > 0$*, depending only on fixed numbers* λ, a *and* n *such that holds*

$$\sup_{S_{m,k}} P_{B_{m,k}}\,(t,x) \leq C_1 \cdot P_{B_{m,k}}\,(0,0)\,, \tag{3}$$

and also such finite partition that at every $j \in \{1,\ldots,n_0\}$

$$\inf_{T^{(j)}_{m,k+1}} P_{B_{m,k}}\,(t,x) \geq C_2 \cdot P_{B_{m,k}}\,(0,0)\,, \tag{4}$$

moreover $C_2 > C_1$.

Lemma 3. *Let bounded domain* $D \subset R^{n+1}$ *containing in cylinder* $C_{m,k}$*, intersecting by cylinder* $C_{m,k+1}$*:* $D \cap C_{m,k+1} \neq \emptyset$ *and* $u(t,x)$ *be a solution of the equation (1) positive in* D*, continuous in* $\overline{D}$ *and vanishing on such part of the parabolic boundary* $\partial_p D$ *of the domain* D*, which lies strongly inside* $C_{m,k}$*. Then for every* $j \in \{1,\ldots,n_0\}$ *we have*

$$\sup_{D\cap T^{(j)}_{m,k}} u(t,x)\ \geq \left(1+\eta P_{D^c\cap T^{(j)}_{m,k}\cap B_{m,k}}\,(0,0)\right) \sup_{D^c\cap T^{(j)}_{m,k+1}} u(t\ ,x), \tag{5}$$

where $\eta > 0$*- absolute constant and* $D^c = \mathbb{R}^{n+1}\diagdown D$.

Now we formulate the main result of this thesis.

Theorem. *A point* $(\mathbf{0},0) \in \partial D$ *is regular with respect to the Dirichlet problem (1) if and only if*

$$\sum_{m,k=1}^{\infty} P_{\boldsymbol{B_{m,k}}}(0,0) = \infty.$$

Corollary. *A point* $(\mathbf{0},0) \in \partial D$ *is regular with respect to the Dirichlet problem (1) if*

$$\sum_{m=1}^{\infty} e^{mn/2} cap\left(D^c \cap A\left(e^{-m}\right)\right) = \infty.$$

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The sufficient condition for the regularity of the boundary point for the parabolic equation in terms of potential

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Consider the Dirichlet problem

$$\sum_{i,k=1}^{n} a_{i,k}(t,x)\, u_{x_i x_k} - u_t(t,x) = 0, \qquad u|_{\partial_p D} = f(t,x)$$

in the bounded domain $D \subset R^{n+1}$ for the parabolic equation.

Let $(t,x) = (t,\ x_1,\ldots,x_n) \in R^{n+1}$ and $s > 0,\ \ \beta > 0$ are the fixed positive numbers. Denote the function type of Weierstrass kernel in R^{n+1} as

$$K_{s,\beta}(t,x) = \begin{cases} t^{-s} exp\left\{-\frac{|x|^2}{4\beta t}\right\},\ t > 0 \\ 0\ , t \le 0 \end{cases}$$

and for $\lambda > 1$ and $m \in N = N \cup \{0\}$ the paraboloids

$$P_m = \{(t,x) \mid |x|^2 \ < \lambda^m \cdot (-t)\ , t < 0\ \}.$$

Let $B_{m,k} = (\overline{P_{m+1} \backslash P_m} \bigcap \left[-t_k; -\ \frac{3}{4}t_k\right]$ for $m,k \in N$, and
$B_{0,\ k}\ = \overline{P_{0,k}} \bigcap \left[-t_k;\ -\frac{3}{4}t_k\right],\ \ t_1 > 0.$

Define the following cylinders $C_{m,k}$ with two parameters m,k

$$C_{m,k} = \{(t,x) \mid\ -t_k < t < 0,\ \ |x| < a\rho_{m,k}\ \},$$

where $a > 0$ is positive real number and $\rho^2_{m,k} = \lambda^m \cdot t_k$ and denote by $S_{m,k}$ the lateral surface of the cylinder $C_{m,k}$.

Let's call

$$T_{m,k} = C_{m,k} \backslash P_m$$

trapezoids of the compliment of paraboloids

$$P_m = \left\{(t,x) \mid\ |x|^2 < -\lambda^m \cdot t\right\}$$

with respect to the cylinders $C_{m,k}$ and denote by $T^{(j)}_{m,k}$, $j \in \{1,2,\ldots,n_0(n)\}$, the minimal finite partition of $T_{m,k}$ such that for which the following inequality

$$|x - \xi| \le |\xi|$$

is fulfilled at $(t,x) \in T^{(j)}_{m,k+1}$ and $(\tau,\xi)\epsilon T^{(j)}_{m,k}$.

Lemma. *There exist absolute constants* $C_1 \ > 0$ *and* $C_2 \ > 0$, *depending only on fixed numbers* $\lambda,\ a,\ s,\ \beta$ *such that holds*

$$\sup_{S_{m,k}} P_{H^{(j)}_{m,k}}\ (t.x) \leq C_1 P_{H^{(j)}_{m,k}}\ (0.0),$$

and also for the finite partition and for all fixed $j \in \{1,\ldots,n_0(n)\}$ *such that*

$$\inf_{T^{(j)}_{m,k+1}}\ P_{H^{(j)}_{m,k}}\,(t,x) \geq C_2\ P_{H^{(j)}_{m,k}}\,(0,0).$$

moreover $C_2 > C_1$, *where*

$$P_B\,(t,x) = \int_B K_{s,\beta}\,(t-\tau, x-\xi)\,\mu(\tau,\xi)$$

the parabolic potential, with the generated kernel of $K_{s,\beta}(t,x)$, B *is a Borel set and* μ *is a Borel measure.*

Theorem. *In order to the boundary point to be regular with respect to the Dirichlet problem sufficiency is that*

$$\sum_{m,k=1}^{\infty} P_{\boldsymbol{B_{m,k}}}\,(0,0) = \infty$$

On basicity of a certain trigonometric system in a Weighted Lebesgue space

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In this work one perturbed system of exponents $1\bigcup\left\{e^{\pm i\lambda_n t}\right\}_{n\in N}$, where $\lambda_n=\sqrt{n^2+\alpha n+\beta}$, $\forall\, n\in N$, is considered. A weighted Lebesgue space $L_{p,w}\left(-\pi,\pi\right)$, $1<p<+\infty$ is considered, where $w:[-\pi,\pi]\to[0,+\infty]$, is a weight function. A sufficient condition for the basicity of this system depending on the parameters $\alpha;\beta\in R$ in $L_{p,w}\left(-\pi,\pi\right)$ is found, when the weight w satisfies the Muckenhoupt condition A_p.

We will say that $\nu\left(\cdot\right)$ satisfies the Muckenhoupt condition (see, e.g. [1]) A_p, if

$$\sup_{I\subset[-\pi,\pi]}\left(\frac{1}{|I|}\int_I\nu\left(t\right)dt\right)\left(\frac{1}{|I|}\int_I\left|\nu\left(t\right)\right|^{-\frac{1}{p-1}}dt\right)^{p-1}<+\infty,$$

holds. We will denote this fact by $\nu\in A_p$.

We will consider the weighted Lebesgue space $L_{p,\nu}\equiv L_{p,\nu}\left(-\pi,\pi\right)$, with a norm

$$\|f\|_{p,\nu}=\left(\int_{-\pi}^{\pi}\left|f\left(t\right)\right|^p\nu\left(t\right)dt\right)^{\frac{1}{p}}.$$

Consider the system of exponents

$$E_\lambda=\left\{e^{i\lambda_n t}\right\}_{n\in Z},$$

where

$$\lambda_0=0,\lambda_n=signn\sqrt{n^2+\alpha\left|n\right|+\beta},n\neq0.$$

The following theorem is proved in exactly the same way as in the work [2].

Theorem 1. *Let $w\in A_p\left(-\pi,\pi\right),1<p<+\infty$. The system of exponents*

$$\left\{e^{i\left(n+\frac{\alpha}{2}signn\right)t}\right\}_{n\in Z},$$

forms a basis for $L_{p,w}\left(-\pi,\pi\right)$ if and only if it is isomorphic to the classical system of exponents $\left\{e^{int}\right\}_{n\in Z}$ in it.

The following theorem is also true.

Theorem 2. *Let the conditions*

$$\left\{w;\ \nu_\alpha^{-p}w\right\}\subset A_p\,\&\,|\alpha|<2\ ,\quad 1<p<+\infty,$$

$$\lambda_n\neq0\,,\ \forall\,n\in Z\ \&\ \lambda_i\neq\lambda_j\ ,\ i\neq j,$$

hold, where

$$\nu_\alpha(t) = \left|\cos\frac{t}{2}\right|^{-\frac{\alpha}{2}}, \quad t \in (-\pi, \pi).$$

Then the system of exponents E_λ *forms an isomorphic basis to* $\left\{e^{int}\right\}$ *in* $L_{p,w}(-\pi,\pi)$.

In particular, the following corollary directly follows from this theorem, which generalizes the result of Yu.A. Kazmin [3] on non-weighted Lebesgue spaces.

Corollary 1. *Let* $w \in A_p(-\pi,\pi)$, $1 < p < +\infty$. *Then the perturbed trigonometric system*

$$1\bigcup\left\{\cos\sqrt{n^2+\beta}\,t;\ \sin\sqrt{n^2+\beta}\,t\right\}_{n\in N},$$

forms an isomorphic basis to the system $1\bigcup\{\cos n\,t;\ \sin n\,t\}_{n\in N}$ *in* $L_{p,w}(-\pi,\pi)$.

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On the estimation of the derivatives of algebraic polynomials in weighted Lebesgue space

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Let $G \subset \mathbb{C}$ be a bounded Jordan region, $L := \partial G$; $w = \Phi(z)$ be a univalent conformal mapping of Ω onto Δ normalized by $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$. For any $R > 1$ denote by $G_R := int\{z : \ |\Phi(z)| = R\}$. Let $P_n(z)$ arbitrary algebraic polynomial of degree not exceeding n, $n \in \mathbb{N}$ and $h\,(z)$ be a generalized Jacobi weight function, defined as follows:

$$h(z) := \prod_{j=1}^{m} |z - z_j|^{\gamma_j}\,,\ \gamma_j > -1,\ j = 1, 2, ..., m\ ,\ \ z \in G_{R_0},\ R_0 > 1.$$

Let $0 < p \leq \infty$. For a rectifiable Jordan curve L, we introduce:

$$\|P_n\|^p_{\mathcal{L}_p(h,L)} := \left(\int\limits_L h(z)\,|P_n(z)|^p\,|dz| \right)^{\frac{1}{p}},\ 0 < p < \infty, \tag{1}$$

$$\|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} |P_n(z)|\,,\ p = \infty.$$

In this work, we study the problem on estimates of the derivatives $|P'_n(z)|$ in bounded $(\overline{G})$ and unbounded $(\mathbb{C} \backslash \overline{G})$ regions with cusps on the boundary L, using weighted $\mathcal{L}_p$-norm, $p > 1$, depending on behavior of the weight function $h\,(z)$.

Behavior at zero of finite differences of Riesz potentials

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Let $g\left(x\right)$ be a function on R^n. For any point $y \in R^n$ define

$$\Delta_y^1 g\left(x\right) = g\left(x+y\right) - g\left(x\right)$$

and by induction

$$\Delta_y^m g\left(x\right) = \Delta_y^1 \left(\Delta_y^{m-1} g\left(x\right)\right).$$

Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be positive numbers. Define quasi-metric on R^n as

$$\rho\left(x,y\right) = \left(\sum_{k=1}^{n} \left|y_i - x_i\right|^{\frac{2}{\lambda_k}}\right)^{\frac{1}{2}}$$

where $x = \left(x_1, x_2, \ldots, x_n\right), \quad y = \left(y_1, y_2, \ldots, y_n\right)$.

Let $\alpha \in \left(0, n\right)$. Define operator

$$I_\alpha f\left(x\right) = \int_{R^n} \rho\left(x,y\right)^{\alpha-n} f\left(y\right) dy. \tag{1}$$

If $\lambda_1, \lambda_2, \ldots \lambda_n = 1$ then $\rho\left(x,y\right)$ turns into the Euclidean metric and $I_\alpha f\left(x\right)$ turns into the classical Riesz potential.

If $f\left(x\right)$ is non-negative and local integrable function then for the existence almost everywhere of the integral in (1) it is necessary and sufficient that

$$\int_{R^n} \left(1 + \rho\left(0,y\right)\right)^{\alpha-n} f\left(y\right) dy < \infty. \tag{2}$$

By $B\left(x,r\right)$ we denote an open ball of radius r and with center x, that is

$$B\left(x,r\right) = \left\{y \in R^n : \rho\left(x,y\right) < r\right\}$$

We also take the denotations

$$\underline{\lambda} = \min\left(\lambda_1, \lambda_2, \ldots \lambda_n\right), \overline{\lambda} = \max\left(\lambda_1, \lambda_2, \ldots \lambda_n\right).$$

Let $\delta > 0$. For the given continuous and nondecreasing function $\varphi : [0, \delta] \to [0, +\infty]$, $\varphi\left(0\right) = 0$ and for set $A \subset R^n$ define the quantity

$$H_g^\varepsilon\left(A\right) = \inf\left\{\sum_{k=1}^{\infty} g\left(r_k\right) A \subset \bigcup_{k=1}^{\infty} B\left(x^{(k)}, r^k\right), r_k < \varepsilon\right\},$$

where $\varepsilon \in (0, \delta)$ and the infimum is taken on all covering of the set A by the balls $\tau\left(x^{(k)}, r_i\right)$ with radius $r_k < \varepsilon$. Consider the $\lim\limits_{\varepsilon \to 0} H_g^{\varepsilon}(A) = H_g(A)$.

$H_g(A)$ is called $g-$ Hausdorff's measure of set A.

Let $p = \frac{n}{\alpha} > 1$ and $\tau > -1$. By $L_{p,\tau}$ we denote a class off all non-negative and measurable functions f, for which

$$\int_{R^n} f^p(y) \ln^{p+\tau}(2 + f(y))\, dy < \infty.$$

Theorem. Let $f \in L_{p,\tau}$ (2) be fulfilled, $m \in \left[1, \frac{\alpha}{\lambda}\right]$ be a natural number and

$$\frac{2}{\lambda_k} \in N or \frac{2}{\lambda_k} > m$$

for any $k = \overline{1, n}$.

If γ is any positive number then

$$\lim_{\rho(0,x) \to 0} \rho(0, x)^{\gamma - \underline{\lambda} m} \Delta_x^m I_\alpha f(z) = 0,$$

for $H_{g_\gamma}-$ almost every z, where

$$g_\gamma(t) = t^{\gamma p} \ln^{\tau+1}\left(2 + \frac{1}{t}\right).$$

On the commutator of Marcinkiewicz integral with rough kernels on generalized weighted Morrey spaces

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We study the boundedness of the commutators of Marcinkiewicz operators $\mu_{\Omega,b}$ with rough kernels $\Omega \in L_s(S^{n-1})$ for some $s \in (1,\infty]$ and BMO function on generalized weighted Morrey spaces $M_{p,\varphi}(w)$. In the case of $b \in BMO(\mathbb{R}^n)$ we find the sufficient conditions on the pair (φ_1, φ_2) with $s' < p < \infty$ and $w \in A_{p/s'}$ or $1 < p < s$ and $w^{1-p'} \in A_{p'/s'}$ which ensures the boundedness of the operators $\mu_{\Omega,b}$ from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$.

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with the normalized Lebesgue measure $d\sigma = d\sigma(x')$. Suppose that Ω satisfies the following conditions.

(i) Ω is a homogeneous function of degree zero on $\mathbb{R}^n$. That is,

$$\Omega(tx) = \Omega(x) \quad \text{for all} \quad t > 0 \ \text{ and } \ x \in \mathbb{R}^n. \tag{1}$$

(ii) Ω has mean zero on S^{n-1}. That is,

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0, \tag{2}$$

where $x' = x/|x|$ for any $x \neq 0$.

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(f)(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy.$$

It is well known that the Littlewood-Paley g-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley g-function. In this paper, we will also consider the com- mutator $\mu_{\Omega,b}$ which is given by the following expression

$$\mu_{\Omega,b} f(x) = \left(\int_0^\infty |F^b_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F^b_{\Omega,t}(x) = \int_{B(x,t)} \frac{\Omega(x-y)}{|x-y|^{n-1}} \left[b(x) - b(y) \right] f(y) dy.$$

In the case of $b \in BMO(\mathbb{R}^n)$ we find the sufficient conditions on the pair (φ_1, φ_2) with $s' < p < \infty$ and $w \in A_{p/s'}$ or $1 < p < s$ and $w^{1-p'} \in A_{p'/s'}$ which ensures the boundedness of the operators $\mu_{\Omega,b}$ from one generalized weighted Morrey space $M_{p,\varphi_1}(w)$ to another $M_{p,\varphi_2}(w)$.

Theorem 1. [1] *Suppose that $b \in BMO(\mathbb{R}^n)$, Ω satisfies the conditions (1), (2) and $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$.*

Let $s' < p < i$, $w \in A_{\frac{p}{s'}}(\mathbb{R}^n)$ and the pair (φ_1, φ_2) satisfy the condition

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) w(B(x,\tau))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \le C\, \varphi_2(x,r),$$

where C does not depend on x and r. Let also, $1 < p < s$, $w^{1-p'} \in A_{p'/s'}$ and the pair (φ_1, φ_2) satisfy the condition

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<\tau<\infty} \varphi_1(x,\tau) \|w\|^{1/p}_{L_{\frac{s}{s-p}}(B(x,\tau))}}{\|w\|^{1/p}_{L_{\frac{s}{s-p}}(B(x,t))}} \frac{dt}{t} \le C\, \varphi_2(x,r) \frac{w(B(x,r))^{\frac{1}{p}}}{\|w\|^{\frac{1}{p}}_{L_{\frac{s}{s-p}}(B(x,r))}},$$

where C does not depend on x and r.

Then the operator $\mu_{\Omega,b}$ is bounded from $M_{p,\varphi_1}(w)$ to $M_{p,\varphi_2}(w)$. Moreover

$$\|\mu_{\Omega,b} f\|_{M_{p,\varphi_2}(w)} \lesssim \|f\|_{M_{p,\varphi_1}(w)}.$$

Remark*Note that Theorem 1. in the case $s = \infty$ was proved in [2].*

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Uniqueness and stability of the solution of the inverse problem for equation of parabolic type

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By given functions $f(x,t,u)$, $\varphi(x)$, $\psi(x,t,u)$, $q(t)$ it is required to determine a pair of functions $\{c(t)\ ,\ u(x,t)\}$ from the conditions:

$$u_t - \Delta u + c(t)\,u = f(x,t,u), \qquad (x,t) \in \Omega = D \times (0,T], \tag{1}$$

$$u(x,0) = \varphi(x),\ x \in \bar{D} = D \bigcup \partial D, D \subset R^n, \tag{2}$$

$$\frac{\partial u}{\partial \nu} = \psi(x,t,u), \quad (x,t) \in S = \partial\, D \times [0,T]\ , \tag{3}$$

$$\int_0^T u(x,t)\,dx = q(t),\ t \in [0,T], 0 < T = const. \tag{4}$$

We assume that the input data of the problem satisfy the following conditions:

1^0.$f(x,t,p) \in C^{\alpha,\alpha/2}\left(\bar{\Omega} \times R^1\right)$, there is a constant $m_1 > 0$, such that for all $p_1, p_2 \in R^1$ and $(x,t) \in \bar{\Omega}$ $|f(x,t,p_1) - f(x,t,p_2)| \le m_1 |p_1 - p_2|$;

$$20.\varphi(x) \in C^{2+\alpha}\left(\bar{D}\right);$$

3^0.$\psi(x,t,p) \in C^{\alpha,\alpha/2}\left(S \times R^1\right)$, there is a constant $m_1 > 0$, such that for all $p_1, p_2 \in R^1$ and $(x,t) \in S$ $|\psi(x,t,p_1) - \psi(x,t,p_2)| \le m_2 |p_1 - p_2|$;

$$40.q(t) \in C^{\alpha}[0,T].$$

Definition 1. *A pair of functions* $\{c(t), u(x,t)\}$ *is called a solution to problem (1)-(4) , if:*

$$1)c(t) \in C[0,T];$$

$$2)u(x,t) \in C^{2,1}(\Omega) \bigcap C^{1,0}\left(\bar{\Omega}\right);$$

3) these functions the relations of the system (1)-(4) are satisfied.

Definition 2. *The set*$K^{\alpha}_{2,2}$ *is called the well-posedness set for the problem (1)-(4) if:*

$$K^{\alpha}_{2,2} = \{(c,u)\,|c(t) \in C^{\alpha}[0,T]\ , \quad u(x,t) \in C^{2+\alpha,1+\alpha/2}\left(\bar{\Omega}\right),$$

there are constants $m_2, m_3 > 0$ *such that*

$$\left|D^l_x u(x,t)\right| \le m_2,\ l = 0,1,2,\ (x,t) \in \bar{\Omega},\ |c(t)| \le m_3,\ t \in [0,T]\right\}.$$

Definition 3. *Let* $\{c_k(t),\ u_k(x,t)\}$ *be solutions of problem (1)-(4) corresponding to the data* $f_k(x,t,u),\ \varphi_k(x),\ \psi_k(x,t,u),\ q_k(t),\ k=1,\,2.$

We say that the solution of problem (1)-(4) is stable if for any $\varepsilon>0$ there is a $\delta(\varepsilon)>0$ such that for $\|f_1-f_2\|_0<\delta,\ \|\varphi_1-\varphi_2\|_2<\delta,\ \|\psi_1-\psi_2\|_0<\delta,\ \|q_1-q_2\|_1<\delta$ the inequality $\|c_1-c_2\|_0+\|u_1-u_2\|_0<\varepsilon.$

The following theorem is proved:

Theorem. *Let*

$$1)\,|q_k\,(t)|\geq const>0,\quad k=1,\ 2,\quad t\in[0,T]\,;$$

2) functions $f_k(x,t,u),\ \ \varphi_k(x),\ \ \psi_k(x,t,u),\ \ q_k(t),\ k=1,\,2,$ *satisfy conditions* 1^0, 2^0, 3^0, 4^0 *respectively;*

3) there exists solutions $\{c_k(t),\ \ u_k(x,t)\}\,,\quad k=1,\,2,$ *to problems (1)-(4) and they belong to the set* $K^{\alpha}_{2,2}$.

Then there exists a $T^*>0$ *such that for* $(x,t)\in\bar{D}\times[0,T^*]$ *a solution for problem (1)-(4) is unique and the following estimate is true:*

$$\|c_1-c_2\|_0+\|u_1-u_2\|_0\leq$$

$$\leq M\left[\|\,f_1-f_2\|_0+\|\,\varphi_1-\varphi_2\|_2+\|\,\psi_1-\psi_2\|_0+\|\,q_1-q_2\|_1\right],$$

where M *depends on the data of problem (1)-(4) and the set* $K^{\alpha}_{2,2}$.

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Using analytical expression of depression curve for study of work of horizontal drain

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During research carried out in the desert and other directions, information dependent on various factors are formed, and between these factors in the form of a table, it is necessary to determine the analytic expression, empiric dependency. Empiric dependencies are found approximately when choosing if it is tried that the sought-for function is the best approach to the function. We can determine the approximation anuel section of these two functions by various methods, the least squares method, averaging method, equalization method. In general, obtaining empiric formulas consists of two stages, clarification of the formula in the general form; the best definition of its parameters and finding analytic expression of the functional corresponding to the physical process. In the general form determination of the dependence by the least squares method is as follows:

$$S\left(a_0, a_1, a_2, \ldots a_n,\right) = \sum_{i=1}^{N} \left(f\left(x_i, a_0, a_1, \ldots a_n\right) - y_i\right)^2 \to min \tag{1}$$

In the paper, at first, we need to construct depression curves based on the information of observation wells located at different distances from the drain space field by means of the least squares method.

The study for selecting the general form of the curve showed that according to the expression (1) the power function $y = ax^b$ is in the form $S\left(a, b\right) = \sum_{i=1}^{n} \left[ax_i^b - y_i\right]^2$ [1,2]. According to our research direction, the analytic expression is in the form $H = aL^b$. Each determined empiric expression was clarifed by the approximation coefficient $R^2 = 1 - \frac{\sum_i (y_i - \widehat{y_i})^2}{\sum_i (y_i - \overline{y})^2}$. All analytic expression were determined being the interval $[0, B/2]$. Here B is the distance between drains and according to the considered study was accepted $B = 200m.H$ is pressure on drain space $(m), L$ is the distance from the drain to the central well (m). The coefficients a and b are determined by using the least squares method.

The construction of the depression curve is based on the observation information carried out on the mode of ground waters after washing and this enables to obtain more exact curves. The distances of wells from the drain are 5, 20, 50, 100 m respectively (fig.).

It is known that the ratio of the analytic between the tangents in the edge points of the arch to the length (α) of the arch is said to be the mean curvature of the arch (l) The mean curvature characterizes the curvature along the whole curved arch. The bending (or curvature) degree in the vicinity of various points of the curve can be different [3].

Therefore, the notion of curvature is given at this point and smoothness of the curvature of the line is taken positive and this time the formula of the curvature is taken in the form of

$$K = \lim_{M_0 \to M} K_{mean} = \left| \lim_{\Delta l \to 0} \frac{\Delta \varphi}{\Delta l} \right| \quad or \; K_{mean} = \left| \frac{d\alpha}{ds} \right| \tag{2}$$

To calculate the curvature of the given curve at the point the formula (2) is used.

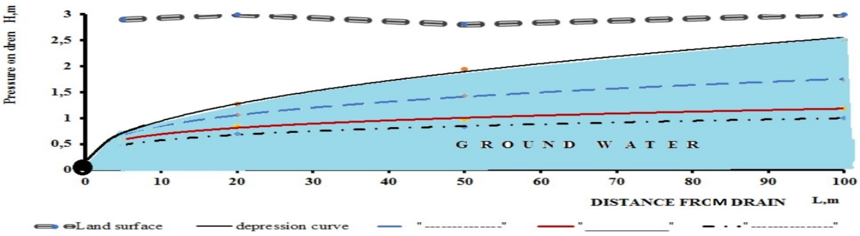

Fig.: Depression curve in interdrain field

The curvature of the curve given by the equation $y = f(f)$ was calculated with regard to the fact that the function has a second-order derivative, the expression of the arch differential, and so on, in the following form. Using the obtained empiric expression, the curvature of the depression curves at the points located at different distances from the drain is calculated by the following formula

$$|K_m| = \frac{H^{//}}{\left[1 + (H)^2\right]^{\frac{3}{2}}} \tag{3}$$

Here $|K_m|$ curvature of the curve at any point in our case the point where the curvature is determined is accepted as 5, 20, 35, 50, 75, 100 m respectively, and is calculated by the formula (3) (table). The expression $H^{/}$ and $H^{//}$ are the first and second derivatives of the curve given by the analytic expressions $H = aL^b$.

Table

Analytic expressions of depression curves in the interdrain field and curvature values

Analytic expressions of depression curves	Calculated values of curvature of depression curves (m) of in different distances from the drain K_m					
B=200 m	5	20	35	50	75	100
$H_1 = 0.351L^{0,43}$ ($R^2 = 0{,}999$)	0,00685	0,00078	0,00033	0,000019	0,00010	0,00006
$H_2 = 0.410L^{0,32}$ ($R^2 = 0{,}999$)	0,00588	0,00057	0,00022	0,000012	0,00006	0,00004
$H_3 = 0.403L^{0,23}$ ($R^2 = 0{,}993$)	0,00421	0,00036	0,00014	0,00007	0,00004	0,00002
$H_4 = 0.329L^{0,24}$ ($R^2 = 0{,}995$)	0,00355	0,00031	0,00012	0,00006	0,00003	0,00002
$H_5 = 0.306L^{0,25}$ ($R^2 = 0{,}982$)	0,00343	0,00030	0,00011	0,00006	0,00003	0,00002

R^2 is called the approximation coefficient.

According to the information given in the table, it was determined that the change depression level on the drain for the highest H_1 the lowest H_5. The analysis of the ratio of curvature at different distances from the drain, according to $K_5/H_{20} \approx 9,5$, $K_{20}/H_{35} \approx 2,5$, $K_{35}/H_{50} \approx 2$, $K_{50}/H_{75} \approx 2$, $K_{75}/H_{100} \approx 2$ was shown that it was highest the distance about 20 m from the drain.

Conclusion

The study of empiric curves obtained based on actual information enables us to conduct assessments in various directions in terms of investigating the work of horizontal drains. The analysis of curvatures in various distances from the interdrain field shows that the curvature in the part 20 m from the drain is the highest at 9,5 in the remaining parts it decreases and equals 2.

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Fundamental solutions of the Cauchy-Goursat type problem for second-order hyperbolic equations and related optimal control problems

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This paper is studied one type of the Cauchy-Goursat problem for a hyperbolic equation of the second order. Under very weak restrictions of the type of boundedness and summability on the coefficients, sufficient conditions are found which this problem is everywhere correctly solvable together with its adjoint system and there exists a unique θ- fundamental solution. Further, with the help of the results obtained, necessary and sufficient optimality conditions are found.

Let $W_p^{(1,1)}(G)$, $1 \le p \le \infty-$ the space of all having generalized derivatives in the sense of S.L. Sobolev $D_x^i D_y^j u \in L_p(G)$, $i = 0,1;\ j = 0,1$, where
$D_z = \frac{\partial}{\partial z}$ [1].

Banach space
$W_p^{(1,1)}(G)$ in the norm

$$||u||_{W_p^{(2,1)}(G)} = \sum_{\substack{0 \le i \le 1 \\ 0 \le j \le 1}} ||D_x^i D_y^j u||_{L_p(G)}$$

Consider the equation

$$\begin{aligned} \left(L^0 u\right)(x,y) = u_{xy}(x,y) + a_{1,0}(x,y)\, u_x(x,y) + \\ + a_{0,1}(x,y)\, u_y(x,y) + a_{0,0}(x,y)\, u(x,y) = \varphi^0(x,y),\ (x,y) \in G, \end{aligned} \tag{1}$$

Where $\varphi^0 \in L_p(G)$

Let us set for equation (1) on some monotone line Γ the conditions

$$\begin{aligned} (L_1 u)(x) \equiv u_X(x,y)\big|_{y=S_\Gamma(x)} = \varphi_1(x),\ x \in G_1 \\ (L_2 u)(y) \equiv u_y(x,y)\big|_{x=\nu_\Gamma(y)} = \varphi_2(y),\ y \in G_2 \\ L_0 u \equiv u(x_0, y_1) = \varphi_0, \end{aligned} \tag{2}$$

where $\varphi_1 \in L_p(G_1)$, $\varphi_2 \in L_p(G_2)\, u\ \varphi_0 \in R$ are given elements.

In this case, the function $E = \nu_\Gamma(y)$ can be considered as a generalized inverse for the function $y = S_\Gamma(x)$ and conversely.

Theorem 1. *Problem (1),(2) has a unique θ fundamental solution θ*
$f(x,y) = (f_0(x,y),\ f_1(\tau,x,y),\ f_2(\xi,x,y),\ f^0(\tau,\xi,x,y))$ from E_q and the solution $u \in W_p^{(1,1)}(G)$ of problem (1),(2) is represented in the form

$$\begin{aligned} u(x,y) = f_0(x,y)\ \varphi_0 + \int_{G_1} f_1(\tau,x,y)\,\varphi_1(\tau)\,d\tau \\ + \int_{G_2} f_2(\xi,x,y)\,\varphi_2(\xi)\,d\xi + \iint_G f^0(\tau,\xi,x,y)\,\varphi^0(\tau,\xi)\,d\tau d\xi. \end{aligned}$$

Assume that the behavior of a managed object is defined by the second-order hyperbolic equation

$$\left(L\overset{0}{u}\right)(x,y)=\varphi^0\left(x,y;\ \ \upsilon^0\left(x,y\right)\right. \tag{4}$$

under conditions

$$\begin{array}{l}(L_1u)\ \ (x)=\varphi_1\left(x,\upsilon_1\left(x\right)\right),\\ (L_2u)\ \ (y)=\varphi_2\left(y,\upsilon_2\left(y\right)\right),\\ L_0u=\varphi_0\left(\upsilon_0\right),\end{array} \tag{5}$$

where $\varphi^0\left(x,y,\upsilon^0\right),\ \varphi_1\left(x,\upsilon_1\left(x\right)\right),\varphi_2\left(y,\upsilon_2\left(y\right)\right)$ and $\varphi_0\left(\upsilon_0\right)$ are given functions defined respectively on $G\times R^r,\ G_1\times R^{r_1},\ G_2\times R^{r_2}$ and R^{r_0}.

$$\upsilon^\circ\left(x,y\right)=\left(\upsilon_1^\circ\left(x,y\right),\upsilon_2^\circ\left(x,y\right),...,\upsilon_r^\circ\left(x,y\right)\right),$$

$$\upsilon_1\left(x\right)=\left(\upsilon_{11}\left(x\right),\upsilon_{12}\left(x\right),...,\upsilon_{1\,r_1}\left(x\right)\right),$$

$$\upsilon_2\left(y\right)=\left(\upsilon_{21}\left(y\right),\upsilon_{22}\left(y\right),...,\upsilon_{2\,r_2}\left(y\right)\right),$$

are control vector functions, and $\upsilon_0=\left(\upsilon_{01},\upsilon_{02},...,\upsilon_{0r_0}\right)$ the control vector

A quadruple of vectors $\left(\upsilon^\circ\left(x,y\right),\upsilon_1\left(x\right),\upsilon_2\left(y\right),\upsilon_0\right)$ is called admissible controls if $\upsilon^0\left(x,y\right),\upsilon_1\left(x\right),\upsilon_2\left(y\right)$ bounded measurable functions with values, respectively, from some given sets $V\subset R^r$, $V_1\subset R^{r_1}$, $V_2\subset R^{r_2}$; $\left(\upsilon^\circ\left(x,y\right)\in V,\ \ \upsilon_1\left(x\right)\in V_1,\upsilon_2\left(x\right)\in V_2\right)$and $\upsilon_\circ\in V_\circ\subset R^{r_0}$.

Assume that $\varphi^0\left(x,y,\upsilon^0\right),\varphi_1\left(x,\upsilon_1\right)\ u\ \varphi_2\left(y,\upsilon_2\right)$on $G\times R^r$, $G_1\times R^{r_1}uG_2\times R^{r_2}$ satisfies the Caratheodory conditions and that for any $\ell>0$ there exist functions $\varphi_e^0\in L_p\left(G\right),\ \varphi_{1,e}^0\in L_p\left(G_1\right),\ \varphi_{2,e}^0\in L_p\left(G_2\right)$ such that $\left|\varphi^0\left(x,y,\upsilon\right)\right|\leq\varphi_e^0\left(x,y\right),\left|\varphi_1\left(x,\upsilon_1\right)\right|\leq\varphi_{1,e}^0\left(x\right),\left|\varphi_2\left(y,\upsilon_2\right)\right|\leq\varphi_{2,e}^0\left(y\right)$ for almost all $(x,y)\in G$ and for all $\left\|\upsilon^0\right\|\leq\ell,\left\|\upsilon_1\right\|\leq\ell,\ \left\|\upsilon_1\right\|\leq\ell,\ \left\|\upsilon_2\right\|\leq\ell$.

Moreover, $\varphi_0\left(\upsilon_0\right)$ is continuous on R^{r_0}. Under the conditions of Theorem 1, problem (3),(4) for any admissible control$\hat{\upsilon}=\left(\upsilon^0\left(x,y\right),\upsilon_1\left(x\right),\upsilon_2\left(y\right),\upsilon_0\right)$ has a unique solution $u\in W_p^{(1,1)}\left(G\right)$. Therefore, we can consider the problem of minimizing the functional

$$S\left(\hat{\upsilon}\right)=A\,u\ \ \left(x_1,y_1\right)$$

to the set of solutions to equation (4) satisfying the given conditions (5).

Let's put $H\left(\tau,\xi,\lambda,\upsilon^0\right)=\lambda\varphi^0\left(\tau,\xi,\upsilon^0\right),\ H_1\left(\tau,\lambda_1,\upsilon_1\right)=\lambda_1\varphi_1\left(\tau,\upsilon_1\right)$, $H_2\left(\xi,\lambda_2,\upsilon_2\right)=\lambda_2\varphi_2\left(\xi,\upsilon_2\right)$ and $H_0\left(\lambda_0,\upsilon_0\right)=\lambda_0\varphi\left(\upsilon_0\right)$ where λ, λ_1, λ_2 and λ_0 are defined as a solution to a system of conjugate equations.

Next, we prove the following

Theorem 2. *For* $\hat{\upsilon}=\left(\upsilon^0\left(x,y\right),\upsilon_1\left(x\right),\upsilon_2\left(y\right),\upsilon_0\right)$*an admissible control to be optimal, it is necessary and sufficient that the following conditions are satisfied:*

$$\begin{array}{l}\max\limits_{\upsilon\in v}H\left(\tau,\xi,\lambda\left(\tau,\xi\right),\upsilon\right)=H\left(\tau,\xi,\lambda\left(\tau,\xi\right),\upsilon\left(\tau,\xi\right)\right),\\ \max\limits_{\upsilon_1\in v_1}H_1\left(\tau,\lambda_1\left(\tau\right),\upsilon_1\right)=H_1\left(\tau,\lambda_1\left(\tau\right),\upsilon_1\left(\tau\right)\right),\\ \max\limits_{\upsilon_2\in v_2}H_2\left(\xi,\lambda_2\left(\xi\right),\upsilon_2\right)=H_2\left(\xi,\lambda_2\left(\xi\right),\upsilon_2\left(\xi\right)\right),\\ \max\limits_{\eta\in\upsilon_0}H_0\left(\lambda_0,\eta\right)=H_0\left(\lambda_0,\upsilon_0\right).\end{array}$$

The totality of these conditions is called the maximum principle of L.S. Pontryagin. Various special cases were previously considered in [2,3,4]

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On bessel property and unconditional basicity of the system of root vector-functions $2m$-th order Dirac type operator with summable coefficient

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Let $L_p^{2m}(G),\ p \geq 1,\ m \geq 1$ be a space 2m-component vector-functions with the norm

$$\|f\|_{p,2m} = \left[\int_G \left(\sum_{j=1}^{2m} |f_j(x)|^{p/2} \right) dx \right]^{1/p}$$

In case $p = \infty,\ \|f\|_{\infty,2} = \sup vrai\ |f(x)|$. For $f(x) \in L_p^{2m}(G),\quad g(x) \in L_q^{2m}(G)$, where $p^{-1} + q^{-1} = 1,\quad 1 \leq p \leq \infty$the scalar product $(f,g) = \int_G \sum\limits_{j=1}^{2m} f_j(x)\overline{g_j(x)}\,dx$ is determined.

Let us consider 2m-th order Dirac type operator

$$Dy = B\frac{dy}{dx} + P(x)\,y,\ y(x) = (\,y_1(x), y_2(x), ..., y_{2m}(x)\,)^T$$

where

$$\begin{aligned} B &= \begin{pmatrix} 0 & 0 & \cdots & 0 & b_1 \\ 0 & 0 & \cdots & b_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{2m} & 0 & \cdots & 0 & 0 \end{pmatrix} (b_1 = b_2 = \cdots = b_m > 0; \\ b_{m+1} &= b_{m+2} = \cdots = b_{2m} < 0) \end{aligned}$$

$P(x) = diag\,(p_1(x), p_2(x), ..., p_{2m}(x)\,)$, moreover $p_i(x),\ i = \overline{1,2m}$ are complex-valued functions determined on arbitrary interval $G = (a,b)$ of real straightline.

Following [1], we will understand root functions of the operator D irrespective to the form of boundary conditions, more exactly, under the eigen-function of the operator D responding to the complex eigen-value λ, we will understand any identically non-zero complex-valued vector-function $\overset{0}{u}(x)$, absolutely continuous an any closed subinterval G and almost everywhere in G satisfies the equation $D\overset{0}{u} = \lambda\overset{0}{u}$.

Similarly, under the self-adjoint function $l,\ l \geq 1$ responding to the same λ and eigen-function $\overset{0}{u}(x)$, we will understand any complex-valued vector-function $\overset{l}{u}(x)$ that is absolutely continuous on any closed subinterval of the interval G and almost everywhere in G satisfies the equation $D\overset{l}{u} = \lambda\overset{l}{u} + \overset{l-1}{u}$.

Theorem 1(Criterian of Bessel property). *Let G be a finite interval and $P(x) \in L_1(G)$, the lengths of the chains of root vector-functions be uniformly bounded and there exists a constant C_0 such that*

$$|Im\lambda_k| \leq C_0, \quad k = 1, 2, \dots . \tag{1}$$

Then for the Bessel property of the system $\left\{ u_k(x) \, \|u_k\|_{2,2m}^{-1} \right\}_{k=1}^{\infty}$ in $L_2^{2m}(G)$ it is necessary and sufficient the existence of a constant M_1 such that

$$\sum_{|Re\lambda_k - \nu| \leq 1} 1 \leq M_1,$$

where ν is an arbitrary real number.

Let $\{\upsilon_k(x)\}_{k=1}^{\infty}$ be a system biorthogonally associated to $\{ u_k(x) \}_{k=1}^{\infty}$ in $L_2^{2m}(G)$ and consists of the root vector-functions of the operator formally associated to the operator D.

Theorem 2 (On unconditional basicity). *Let G be a finite interval and $P(x) \in L_1(G)$, the lengths of the systems$\{u_k(x)\}_{k=1}^{\infty}$ and $\{\upsilon_k(x)\}_{k=1}^{\infty}$ be complete in $L_2^{2m}(G)$ and condition (1) be fulfilled.*

Then necessary and sufficient condition for unconditional basicity in $L_2^{2m}(G)$ of each of these systems is the existence of constants M_1 and M_2 that ensure validity of inequality (2) and

$$\|u_k\|_{2,2m} \, \|\upsilon_k\|_{2,2m} \leq M_2, \; k = 1, 2, \dots .$$

Note that under the conditions of theorem 2 satisfaction of inequalities (2) and (3) is necessary and sufficient for Riesz basicity of each of the systems

$$\left\{ u_k(x) \, \|u_k\|_{2,2m}^{-1} \right\}_{k=1}^{\infty} \text{ and } \left\{ \upsilon_k(x) \, \|\upsilon_k\|_{2,2m}^{-1} \right\}_{k=1}^{\infty} \text{ in } L_2^{2m}(G).$$

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Commutators of multilinear Calderón-Zygmund operators with kernels of Dini's type on generalized local Morrey spaces

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The study of multilinear Calderón-Zygmund theory goes back to the pioneering works of Coifman and Meyer in 1970s, see e.g. [1]. The classical Morrey spaces $M_{p,\lambda}$ were introduced by Morrey to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. The first author, Mizuhara and Nakai introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$.

Let T be a multilinear Calderon-Zygmund operator such that

$$T(f_1,\dots,f_m)(x)=\int_{(\mathbb{R}^n)^m} K(x,\vec{y})f_1(y_1)\dots f_m(y_m)d\vec{y},$$

$x\notin \bigcap\limits_{j=1}^{m} suppf_j$, where $K(x,\vec{y})$ is an m-linear Calderon-Zygmund kernel of type $\omega(t)$ and each $f_j\in C_c^\infty(\mathbb{R}^n)$, $j=1,\dots,m$. Here and in what follows, $\vec{y}=(y_1,\dots,y_m)$, $(x,\vec{y})=(x,y_1,\dots,y_m)$ and $d\vec{y}=dy_1\dots dy_m$.

Let T be a multilinear Calderón-Zygmund operator of type ω with $\omega(t)$ being nondecreasing and satisfying a kind of Dini's type condition and $T_{\Pi\vec{b}}$ be the iterated commutators of the operator T with BMO^m functions. For an m-linear Calderón-Zygmund operator with associated kernel $K(x,\vec{y})$, the iterated commutator $T_{\Pi\vec{b}}$ is given formally by

$$T_{\Pi\vec{b}}(\vec{f})(x)=\int_{(\mathbb{R}^n)^m}\Big(\prod_{j=1}^{m}(b_j(x)-b_j(y_j))\Big)K(x,\vec{y})f_1(y_1)\dots f_m(y_m)d\vec{y}.$$

In [4], the boundedness of the multilinear Calderon-Zygmund operator T and iterated commutators of the operator T from $M^{\{x_0\}}_{p_1,\varphi_{11}}(\mathbb{R}^n)\times\dots\times M^{\{x_0\}}_{p_m,\varphi_{1m}}(\mathbb{R}^n)$ to $M^{\{x_0\}}_{p,\varphi_2}(\mathbb{R}^n)$ for $\vec{b}=(b_1,\dots,b_m)\in BMO^m$ was proved.

In [2] and [3] Guliyev defined the generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ and the generalized local Morrey spaces $M^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ as follows.

Definition 1. *Let $1\le p<\infty$ and φ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$. We denote by $M^{\{x_0\}}_{p,\varphi}(\mathbb{R}^n)$ and $M_{p,\varphi}(\mathbb{R}^n)$ the generalized local Morrey space, the generalized Morrey space respectively, the spaces of all functions $f\in L_p^{\rm loc}(\mathbb{R}^n)$ with finite norms*

$$\|f\|_{M^{\{x_0\}}_{p,\varphi}}=\sup_{r>0}\varphi(x_0,r)^{-1}\,|B(x_0,r)|^{-\frac{1}{p}}\,\|f\|_{L_p(B(x_0,r))},$$

$$\|f\|_{M_{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \, |B(x,r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))},$$

respectively.

The definition of local Campanato space as follows.

Definition 2. *Let* $1 \le q < \infty$. *A function* $f \in L_q^{\rm loc}(\mathbb{R}^n)$ *is said to belong to the* $CBMO_q^{\{x_0\}}(\mathbb{R}^n)$ *(local Campanato space), if*

$$\|f\|_{CBMO_q^{\{x_0\}}} = \sup_{r>0} \Big(\frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} |f(y) - f_{B(x_0,r)}|^q dy \Big)^{1/q} < \infty,$$

where

$$f_{B(x_0,r)} = \frac{1}{|B(x_0,r)|} \int_{B(x_0,r)} f(y)dy.$$

Define

$$CBMO_q^{\{x_0\}}(\mathbb{R}^n) = \{ f \in L_q^{\rm loc}(\mathbb{R}^n) \ : \ \|f\|_{CBMO_q^{\{x_0\}}} < \infty \}.$$

We study the boundedness of the operators $T_{\Pi\vec{b}}$ on generalized local Morrey spaces $M_{p,\varphi}^{\{x_0\}}(\mathbb{R}^n)$. We find the sufficient conditions on the pair $(\vec{\varphi}_1, \varphi_2)$ with $\vec{b} \in CBMO_{\vec{q}}^{\{x_0\}}$ which ensures the boundedness of the operators $T_{\Pi\vec{b}}$ from $M_{p_1,\varphi_{11}}^{\{x_0\}}(\mathbb{R}^n) \times \cdots \times M_{p_m,\varphi_{1m}}^{\{x_0\}}(\mathbb{R}^n)$ to $M_{p,\varphi_2}^{\{x_0\}}(\mathbb{R}^n)$, where $CBMO_{\vec{q}}^{\{x_0\}} \equiv CBMO_{q_1}^{\{x_0\}} \times \cdots \times CBMO_{q_m}^{\{x_0\}}$.

Our main results formulated as follows:

Theorem. Let $m \ge 2$, $x_0 \in \mathbb{R}^n$, T be an m-linear $\omega-CZO$ and ω satisfies (1). Suppose that $\vec{b} = (b_1, \ldots, b_m) \in CBMO_{q_1}^{\{x_0\}} \times \cdots \times CBMO_{q_m}^{\{x_0\}}$, $(p_1, \ldots, p_m)$, $(q_1, \ldots, q_m) \in (1,\infty)^m$, $p \in (0,\infty)$ with $1/p = \sum\limits_{i=1}^m 1/p_i + \sum\limits_{i=1}^m 1/q_i$, and $(\vec{\varphi}_1, \varphi_2)$ satisfies the condition

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)^m \, t^{-\frac{n}{p}}_{t<s<\infty} s^{\frac{n}{p}} \prod_{i=1}^m \varphi_{1i}(x_0,s) \, \frac{dt}{t} \le C \, \varphi_2(x_0,r),$$

where C does not depend on r.

Then there exist constants $C > 0$ independent of $\vec{f} = (f_1, \ldots, f_m)$ such that

$$\|T_{\Pi\vec{b}}(\vec{f})\|_{M_{p,\varphi_2}^{\{x_0\}}} \le C \, \|\vec{b}\|_{CBMO_{\vec{q}}^{\{x_0\}}} \prod_{i=1}^m \|f_i\|_{M_{p_i,\varphi_{1i}}^{\{x_0\}}},$$

where $\|\vec{b}\|_{CBMO_{\vec{q}}^{\{x_0\}}} := \prod\limits_{i=1}^m \|b_i\|_{CBMO_{q_i}^{\{x_0\}}}$.

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On a defective property of a weighted system of sines in Lebesgue spaces

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In [1], it is proved that the system $\{te^{int}\}_{n\in Z\setminus\{k_0\}}$ is complete and minimal, but does not form a basis in $L_p(-\pi,\pi)$. Subsequently, in [2], it was proved that weighted systems of exponents $\{\rho(t)e^{int}\}_{n\in Z\setminus\{k_0\}}$ and weighted trigonometric systems of sines $\{\rho(t)\sin nt\}_{n\in N\setminus\{k_0\}}$ and cosines $\{\rho(t)\cos nt\}_{n\in Z_+\setminus\{k_0\}}$ with a power-law weight function with an arbitrary element removed do not form a basis $L_p(-\pi,\pi)$ in and $L_p(0,\pi)$, respectively, and sufficient conditions were found for their completeness and minimality. These questions in the case of a weight function of a general form $\omega(t)$ were studied in [3, 4]. Note that in [3, 4] criteria for the completeness and minimality of the systems $\{\omega(t)e^{int}\}_{n\in Z\setminus\{k_0\}}$ and $\{\omega(t)\cos nt\}_{n\in Z_+\setminus\{k_0\}}$ in the corresponding spaces $L_p(-\pi,\pi)$ and $L_p(0,\pi)$ were obtained. In this paper, following the ideas of [2, 3], we study the criteria for the completeness and minimality of the system

$$\{\omega(t)\sin nt\}_{n\in N\setminus\{k_0\}}, \tag{1}$$

where $\omega(t)$ is some measurable function on $(0,\pi)$. The following statements are true.

Theorem 1. *System (1) is complete in $L_p(0,\pi), p>1$, if and only if conditions $t(t-\pi)\omega(t)\in L_p(0,\pi)$ and $\frac{t(t-\pi)}{\omega(t)}\notin L_q(0,\pi)$ are satisfied, where $q=\frac{p}{p-1}$.*

Theorem 2. *System (1) is minimal in $L_p(0,\pi), p>1$, if and only if the conditions*

a) $t(t-\pi)\omega(t)\in L_p(0,\pi)$, $\frac{t(t-\pi)}{\omega(t)}\in L_q(0,\pi)$, where $q=\frac{p}{p-1}$ or

b) $t(t-\pi)\omega(t)\in L_p(0,\pi)$, $\frac{t(t-\pi)}{\omega(t)}\notin L_q(0,\pi)$, and one of the conditions holds $\frac{t^3(t-\pi)}{\omega(t)}\in L_q(0,\pi)$ or $\frac{t(t-\pi)^3}{\omega(t)}\in L_q(0,\pi)$ or $\exists t_0\in(0,\pi):\sin k_0t_0\neq 0$ and $\frac{t(t-\pi)(t-t_0)}{\omega(t)}\in L_q(0,\pi)$.

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On the bounds of the number of sheets of covering

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The surjective continuous map π: X$\rightarrow$ Y of linearly connected space X is called the covering:

1) if for every point $a \in Y$ there exists a neighborhood $V \subset Y$ for which it can be found a homeomorphism $h:\ \pi^{-1}(V) \rightarrow V \times \Gamma$, with discrete space Γ.

2) If $p: V \times \Gamma \rightarrow V$ is a natural projection then

$$\pi|_{\pi^{-1}(V)} = p^{\circ} h.$$

The space X is called the space of covering, Y is called the base of covering.

In many questions of geometry, it arises the question on defining or estimating of the number of sheets.

Lemma 1. *Suppose we are given with normed spaces X, Y and Z. Let $\Phi: X \times Y \rightarrow Z$ be some differentiable map which at the points $a \in X$, $b \in Y$ satisfies the condition $\Phi(a,b) = 0$ and the linear operator $\frac{\partial \Phi(x,y)}{\partial y}$ is continuous and has an inverse in some neighborhood U of the point (a,b). Then there exists such a neighborhood $W = \{(x,y)|\, \|x-a\| < r,\ \|y-b\| < \rho\} \subset U$ of this point and a unique mapf $: X \rightarrow Y$ such that:*

1) f is continuous in the ball $U_r = \{x|\, \|x-a\| < r\}$;

2) the equality $b = f(a)$ is true;

3) for every $x \in U_r$ all of pairs (x,y) in the neighborhood W satisfying the equality $\Phi(x,y) = 0$ are given by the equality $y = f(x)$.

Lemma 2. *Suppose that in a bounded and closed domain W of n-dimensional space they are given some continuous function $f(\bar{x}) = f(x_1, ..., x_n)$ and continuousely differentiable funcions $f(\bar{x}) = f(x_1, ..., x_n)$ $j = 1, ..., r$, $r < n$, with non-degenerating Jacoby matrix*

$$\frac{\partial(f_1, ..., f_r)}{\partial(x_1, ..., x_n)}.$$

Let $\bar{\xi}_0 = (\xi_1^0, ..., \xi_r^0)$ be such an inner point of the image of map F: $\overline{x} \rightarrow (f_1, ..., f_r)$. Then in some neighborhood of the point $\bar{\xi}_0$ the following relation holds true:

$$\frac{\partial^r}{\partial \xi_1 \cdots \partial \xi_r} \int_{\Omega(\hat{\xi})} f(\hat{x}) d\hat{x} = \int_{M(\xi)} f(\hat{x}) \frac{ds}{\sqrt{G}};$$

where $\Omega(\bar{\xi})$ is a subdomain of Ω, defined by conditions $f_j(\bar{x}) \le \xi_j$, $M(\bar{\xi})$ is a surface, defined by the system of equations $f_j(\bar{x}) = \xi_j (j = 1, ..., r)$, and G is a Gram determinant of the system of functions, that is $G = |(\nabla f_i, \nabla f_j)|$.

Let us formulate now the theorem on implicit functions in $\mathbb{R}^n$ for the system of equations.

Lemma 3. *Suppose the conditions of Lemma 2 is satisfied. Consider the conditions:*
1) There exists a point $\left(\hat{a},\hat{b}\right)=(a_1,...,a_{n-r},b_1,...,b_r)$ *for which*

$$\left\{\begin{array}{c} f_1(a_1,...,a_{n-r},b_1,...,b_r)=0, \\ \cdots \\ f_r(a_1,...,a_{n-r},b_1,...,b_r)=0. \end{array}\right\};$$

2) There exists a neighborhood W *of the point* $\left(\bar{a},\bar{b}\right)$ *in which the functions above have continuous partial derivatives*

$$\frac{\partial f_i\left(x_1,...,x_{n-r},y_1,...,y_r\right)}{\partial y_j},\ i=1,...,r;?\ j=1,...,n-r.$$

3) In the specified neighborhood of the point $\left(\bar{a},\bar{b}\right)$ *following determinant is distinct from zero:*

$$J(\overline{y})=\frac{\partial f(\hat{x},\hat{y})}{\partial\hat{y}}=\left(\begin{array}{ccc} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_r} \\ \cdots & \ddots & \cdots \\ \frac{\partial f_r}{\partial y_1} & \cdots & \frac{\partial f_r}{\partial y_r} \end{array}\right).$$

Then there exists a neighborhood U_δ *of the point* $\hat{a}=(a_1,...,a_{n-r})\in R^{n-r}$ *and the unique system of continuous functions*

$$\left\{\begin{array}{c} y_1=y_1(x_1,...,x_{n-r}), \\ \cdots\cdots\cdots \\ y_r=y_r(x_1,...,x_{n-r}) \end{array}\right\}$$

for which

$$\left\{\begin{array}{c} y_1(a_1,...,a_{n-r})=b_1, \\ \cdots\cdots\cdots \\ y_r(a_1,...,a_{n-r})=b_r, \end{array}\right\}$$

and in neighborhood U_δ *identically satisfied the following equalities:*

$$\left\{\begin{array}{c} f_1(x_1,...,x_{n-r},y_1\left(x_1,...,x_{n-r}\right),...,y_r\left(x_1,...,x_{n-r}\right))\equiv 0, \\ \cdots\cdots\cdots \\ f_{m-r}(x_1,...,x_{n-r},y_1\left(x_1,...,x_{n-r}\right),...,y_r\left(x_1,...,x_{n-r}\right))\equiv 0. \end{array}\right\}$$

Consider the manifold defined by Lemma 3.

Theorem. *Takes some cube* $B\subset R^{n-r}$ *such that covering with this base has one and the same discrete space* Γ*. Then for the number of elements of* Γ *the following inequality is satisfied:*

$$|\Gamma|\leq \mathrm{M}|\mathrm{B}|^{-1}\int_{\Pi}\frac{ds}{\sqrt{G}},$$

where M *denotes the maximal value of the minors of* $J(\overline{y})$ *in* B *and* $|B|$ *is a volume of* B*.*

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Functionally complete algebras in congruence varieties

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The range of problems studied in the theory of varieties of algebraic systems is very wide, and one of the most important directions included in this circle is the study of functional completeness problems in varieties that satisfy the congruence condition [1]. A new stage for the development of this problem was proposed by I. Rosenberg. When $k \geq 4$, the problem of completeness in k-valued logic ($k = 2, 3$ cases were given by E. Post and S.V. Yablonsky) was given a theorem [2].

Definition 1. If an algebra has no distinct congruence with a common adjoint class, then this algebra is called congruence regular. If every algebra included in a variety has the congruence regularity property, this variety is called congruence regularity (CR).

Chakan [3] proved that a necessary and sufficient condition for the variety, $var\,(\mathcal{A}) \in CR$ to be congruence regular is the presence of three local terms $\mu_1, ..., \mu_n$ such that in this algebra $\mathcal{A}$.

(1) $\mu_i(x, y, z) = z\ (i = 1, ..., n)$

identity and

(2) $\mu_1\,(x, y, z) = z \wedge \cdots \wedge \mu_n\,(x, y, z) = z \to x = y$

be quasi-equally true.

Theorem 1. Relator R_1 is eliminable in congruence regular varieties.

Proof. On the contrary, let us assume that $Pol\,(\mathcal{A}) \subseteq St\,(\leq)$. In accordance with 0 and 1, denote the smallest and the largest element in the algebra $\mathcal{A}$ according to the ratio $\leq$ (since $\leq$ is bounded, there are such elements). Then (2) is quasi-identical to $\mathcal{A}$ A because it is correct in algebra. There is an index $j \in 1, \ldots, n$ such that $\mu_j\,(0, 1, 1) \neq 0$ and let us denote by $\mu_j\,(0, 1, 1) = a$ (here $a \neq 1$). On the other hand, since the identity (1) is true in algebra A, we get $\mu_j\,(0, 1, 1) = 1$.

If we use $0 \leq 0, 1 \leq 1, 1 \leq 1$ and $Term\,(\mathcal{A}) \in Pol\,(\mathcal{A}) \subseteq St(\leq)$ $\mu_j\,(0, 0, 1) \leq \mu_j\,(0, 1, 1)$ is obtained . Since $\mu_j\,(0, 1, 1) = a$, $a \geq 1$, it is clear that $a \leq 1$. From here, $a = 1$ is obtained. The contradiction is obtained. The theorem is proved.

It is obtained from Theorem 1 and [4]

Theorem 2. $ROS_{fc}(CR) = \{R_3, R_4, \hat{R}_5\}$

$ROS_{fc}\,(L)$ let's look at the class here $L \in \{\Gamma_2\,(K)\,; K \in \Phi\}$

$$\Gamma_2\,(CM) = CP, \Gamma_2\,(CD) = AR = CP \cap CD$$

$$\Gamma_2\,(REDPC) = CP \cap REDPC$$

$$\Gamma_2^{'}\,(CD) = \Delta_2, \Gamma_2^{'}\,(FI) = \Gamma_2^{'} = DD$$

DD D the difference is included in the congruence distributive varieties.

Since AR -arithmetic varieties are both congruence distributive, then by the Mackenzie-Gumm-Sandy theorem and Theorem 2.

Corollary 1. $ROS_{fc}(AR) = \{R_4\}$

Corollary 2. $ROS_{fc}\,(CP \cap PEDP) = ROS_{fc}\,(D) = \{R_4\}$

Corollary 3. $ROS_{fc}\,(\Gamma_2\,(CM)) = \{R_3, R_4\}$

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A numerical scheme for a class of distributed-order fractional integro-differential equations

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Fractional order differential equations were mainly confined to theoretical study by mathematicians because of their practical implications were not known in the real world. However, in modern society, the practical results from mathematical models with fractional derivatives have been exceedingly impressive. In particular instances, the fractional derivative has resulted in more enlightening results than the traditional derivative [1].

In most applications, the order of the fractional derivative is allowed to vary within predetermined restrictions, this means a fractional derivative model is more complex than the integer one. Thus, the solution procedure of an equation with a fractional derivative is more laborious.

Fractional differential equations (FDEs) are the special case of distributed-order fractional differential equations (DFDEs) [2]. Thus, models based on the DFDEs became a growing research area due to their potential for describing complex phenomena and several problems in different fields have been modeled via these types of equations [3, 4, 5, 6].

It is of vital importance that the methods of solutions for differential equations be accurate, easy to use and computationally inexpensive. This is crucial so as to avoid misinterpretations or getting results that are inconsistent with the purpose of the mathematical model. There are various methods that have been suggested to obtain an approximate solution of fractional differential equations (FDEs), these include, the use of polynomials [7, 8], orthogonal polynomials [6, 9] and Adams-Bashforth scheme [10].

In this manuscript, we seek to make a contribution toward the numerical solutions of differential equations.

In particular, we want to approximate the solution of the distributed-order integro-differential equation given as[3],

$$\int_0^1 N_1\left(\zeta, {}^{C}D_t^{\alpha} f(t)\right) d\alpha + N_2\left(t, f(t), I_F(f(t)), I_V(f(t))\right) = g(t), \tag{1}$$

where $t \in [0,1]$ and N_1, N_2 are nonlinear operators and

$$\begin{aligned} I_F(f(t)) &= \textstyle\int_0^1 K_1(t,\tau)\, \mu_1\left(f(\tau)\right)\, d\tau, \\ I_V(f(t)) &= \textstyle\int_0^t K_2(t,\tau)\, \mu_2\left(f(\tau)\right)\, d\tau, \end{aligned} \tag{2}$$

subjected to the initial condition

$$f(0) = f_0. \tag{3}$$

In (1), ${}^{C}\mathcal{D}_t^{\alpha}$ is fractional order derivative in the Caputo sense which is defined as follows:
Definition 1. *(See [1]) Suppose $0 < \alpha \leq 1$ and $h \in C[0,1]$, the α-order Caputo derivative is given by*

$$ {}^{C}\mathcal{D}_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau)\, d\tau, & 0 < \alpha < 1, \\ f'(t), & \alpha = 1. \end{cases} $$

In this study, a numerical algorithm is presented to obtain an approximate solution of distributed-order integro-differential equations (1). The approximate solution is expressed in the form of a polynomial with unknown coefficients and in place of differential and integral operators, we make use of matrices that we deduce from the shifted Legendre polynomials. To compute the numerical values of the polynomial coefficients, we set up a system of equations that tallies with the number of unknowns, we achieve this goal through the Legendre–Gauss quadrature formula and the collocation technique. The theoretical aspects of the error bound are discussed. Illustrative examples are included to demonstrate the validity and applicability of the method.

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Backward behavior and determining functionals for chevron pattern equations

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This is joint work with Varga Kalantarov (Koç University, Turkey) and Orestis Vantzos (Vantzos Research SMPC, Greece).

The following system of equations was proposed by Rossberg et al. [4],[5] to model the formation and evolution of patterns in electroconvection (sometimes accompanied with magnetic field) of nematic liquid crystals:

$$\tau\partial_t A = A + \Delta A - \phi^2 A - |A|^2 A - 2ic_1\phi\partial_y A + i\beta A\partial_y\phi, \tag{1}$$

$$\partial_t\phi = D_1\partial_x^2\phi + D_2\partial_y^2\phi - h\phi + \phi|A|^2 - c_2\mathrm{Im}\left[A^*\partial_y A\right], \tag{2}$$

where $D_1 > 0,\ D_2 > 0,\ c_1 \geq 0,\ c_2 \geq 0,\ h \geq 0,\ \beta \in \mathbb{R}$ are given parameters. Here the complex-valued function A, where A^* denotes its complex conjugate, and the real-valued function ϕ are unknown functions.

One can find numerical studies of this mathematical model in the literature [3],[6] . It is also interesting to study it from a mathematical analysis perspective: we have proved in our previous works [1], [2] that the system (1)-(2) in a bounded domain under homogeneous Dirichlet boundary conditions possesses an exponential attractor. We also proved the global stabilization of the zero state of the system by finitely many Fourier modes. Finally, we showed the stabilization of an arbitrary fixed solution in one spatial dimension along with relevant numerical results.

In this work, we study the backward behavior of solutions of the initial boundary value problem qualitatively as well as numerically. We then prove that the asymptotic behavior as $t \to \infty$ of solutions is completely determined by finitely many functionals.

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Blow up of solutions to coupled systems of nonlinear equations

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We studied the problem of blow-up of solutions of the following system of equations

$$u_{tt} + bu_t - \Delta u = f_1(u,v), \tag{1}$$

$$v_t - \Delta v = f_2(u,v), \tag{2}$$

under the initial and boundary conditions

$$u\Big|_{t=0} = u_0(x), \quad u_t\Big|_{t=0} = u_1(x), \quad v\Big|_{t=0} = v_0(x), \quad x \in \Omega, \tag{3}$$

$$u\Big|_{\partial\Omega} = v\Big|_{\partial\Omega} = 0, \tag{4}$$

where b is a given nonnegative number, $\Omega \subset R^n$ is a bounded or unbounded domain with sufficiently smooth boundary $\partial\Omega$, the vector field $F(u,v) := [f_1(u,v), f_2(u,v)]$ is continuous and conservative in R^2, i.e. there exists a differentiable function

$$F(u,v) : R^2 \to R \ \text{ such that } \ \nabla F(u,v) = [f_1(u,v), f_2(u,v)].$$

We assume that

$$uf_1(u,v) + vf_2(u,v) \geq 2(2\alpha+1)F(u,v) - D_0, \quad \forall u,v \in R$$

with some $\alpha > 0$, $D_0 \geq 0$. We found sufficient conditions on the data that guarantee blow up in a finite time of solutions with an arbitrary positive initial energy of the problem (1)-(4).

Stablizaition of solutions to Navier- Stokes-Voigt and convective Brinkman-Forchheimer equations

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The talk is devoted to the problem of global exponential stabilization of solutions to the Navier-Stokes-Voigt equations

$$\begin{cases} v_t - \nu\triangle v - \alpha^2\triangle v_t + (v\cdot\nabla)v + \nabla p = -\mu w, & x\in\Omega, t>0, \\ \nabla\cdot v = 0, \quad x\in\Omega,\ t>0 \end{cases} \tag{1}$$

and the convective Brinkman-Forchheimer equations

$$\begin{cases} v_t - \nu\triangle v + (v\cdot\nabla)v + |v|^m v + \nabla p = -\mu w, & x\in\Omega, t>0, \\ \nabla\cdot v = 0, \quad x\in\Omega,\ t>0, \end{cases} \tag{2}$$

where $\alpha > 0, \nu > 0, \mu > 0, m > 2$ are positive parameters, $\Omega \subset \mathbb{R}^3$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ and w is a feedback control input (different for different problems).
We show that any arbitrary given solution of the initial boundary value problem for each of equations (1) and (2) can be stabilized by using feedback controllers depending only on finitely many large spatial-scale parameters and by controllers acting on a bounded subdomain $\omega \in \Omega$.

On some properties of the Riesz potential in the Grand Lebesgue space

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The work considers the Riesz-type potential in non-standard grand-Lebesgue space. The classical facts about Lebesgue space carry over to this case.

First, define the grand Lebesgue space. R^n is n-dimensional Euclidean space and p' means the conjugate of p, i.e. $\frac{1}{p}+\frac{1}{p'}=1$. Let $\Omega\subset R^n$ be a bounded domain with Lebesgue measure $|\Omega|$. The Grand Lebesgue space $L_{p)}(\Omega)$, $1<p<+\infty$, is the Banach space of measurable functions (according to Lebesgue) on Ω with the norm

$$\|f\|_{p)}=\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{|\Omega|}\int_\Omega |f|^{p-\varepsilon}\,dx\right)^{\frac{1}{p-\varepsilon}}.$$

This space is not separable. We have the following continuous embeddings

$$L_q(\Omega)\subset L_{q)}(\Omega)\subset L_{q-\varepsilon}(\Omega)\ ,\ \ \forall\varepsilon\in(0,q-1)\,.$$

Regarding these and other facts, more details can be found in the works [1-5].

Let $\Omega\subset R^n$ be a bounded domain, $A(\cdot;\cdot)\in C\left(\bar\Omega\times\bar\Omega\right)$ be a continuous function. Consider the following Riesz-type integral

$$(K\rho)(x)=u(x)=\int_\Omega\frac{A(x;y)}{r^\lambda}\rho(y)\,dy, \qquad (1)$$

where $r=|x-y|$ is the distance between points $x;y\in\bar\Omega$, $\lambda\in[0,n)$ is some number. We studied some properties of this integral with respect to the space $L_{q)}(\Omega)$.

The following theorem is true.

Theorem 1. *Let $\Omega\subset R^n$ be a bounded domain, $\rho\in L_{q)}(\Omega)$, $A\in C\left(\overline{\Omega}\times\overline{\Omega}\right)$ and $\lambda q'<n$. Then the operator (1) acts completely continuously from $L_{q)}(\Omega)$ to $C\left(\overline{\Omega}\right)$.*

The following theorem is also true.

Theorem 2. *Let $\lambda q^{'}\geq n$ and an integer s satisfy $n-(n-\lambda)q<s\leq n$. Then the integral (1) defines a function that, on any section Ω_s of the set Ω by a plane of dimension s is defined almost everywhere in the sense of the Lebesgue measure in R^s. The operator K, defined by formula (1) is bounded as an operator from $L_{q)}(\Omega)$ to $L_r(\Omega_s)$ (and hence to $L_{r)}(\Omega_s)$), for $\forall r$: $1<r<r_0=\frac{sq}{n-(n-\lambda)q}$.*

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On the law of large numbers for the of Markov random walks described by the autoregressive process

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It is well known that the first-order autoregressive process ($AR(1)$) is determined by the solution of a recurrent equation of the form

$$X_n = \beta X_{n-1} + \xi_n \tag{1}$$

where $n \geq 1, \beta \in R = (-\infty, \infty)$ is some fixed number and the innovation $\{\xi_n\}$ is the sequence of independent identically distributed random variables with finite variance $G^2 = D\xi_1 < \infty$ and with mean $a = E_1\xi_1$. It is assumed that the initial value of the process X_0 is independent on the innovation $\{\xi_n\}$.

The process $AR(1)$ plays a great role in theoretical and applied terms in the problems of the theory of random walks ([1] - [2]).

The following Markov random walks are described by means of the process $AR(1)$

$$S_n = \sum_{k=0}^{n} X_k,$$

$$C_n = \sum_{k=1}^{n} X_k X_{k-1},$$

$$D_n = \sum_{k=1}^{n} X_{k-1}^2,$$

$$\theta_n = \frac{C_n}{D_n},$$

$$Z_n = \frac{C_n^2}{D_n},$$

$$H_n = \sum_{k=1}^{n} X_{k-1}\xi_x, n \geq 1$$

These Markov random walks have been considered in the works studying some problems of the theory of nonlinear renewal theory. The law of large numbers for the Markov random walk C_n, D_n, $\theta_n n \geq 1$ is proved in for the case $a = 0$ in [1].

In the present paper, we prove the law of large numbers for the mentioned Markov random walks in general cases [2].

Corollary 1. *Let the conditions of the theorem are satisfied, and $a = 0$, then*

1)

$$\theta = \frac{C_n}{D_n} \xrightarrow{P} \beta, \ \ as \ \ n \to \infty,$$

2)

$$\frac{Z_n}{n} \xrightarrow{P} \frac{\sigma^2 \beta^2}{1 - \beta^2}, \ \ as \ \ n \to \infty.$$

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Determining the coefficient at the lowest terms in the system of elliptic equations

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The goal of this paper is to study the well-posedness of an inverse problem of determining the unknown coefficient of the system of elliptic equations of reaction-diffusion type. The uniqueness theorem for the solution of the considered inverse problem is proved, and the "conditional" stability estimate is obtained.
Let $B \subset R^n$be a bounded, strictly convex domain with a sufficiently smooth boundary $\partial B, x = (x_1, ..., x_n) = (x', x_n), B^{'} \subset R^{n-1}$- be the projection of the domain B onto hyperplane $x_n = 0$, $B = B^{'} \times (\gamma^1(x'),\gamma^2(x'))$, where $\gamma^1(x')$ and $\gamma^2(x')$ are given functions.$u = (u_1, ..., u_m)$, $\|u\|_l = \sum_{k=1}^m \|u_k\|_{C^l} = \sum_{k=1}^m \sum\limits_{i=0}^{l} \sup\limits_B \left|D_x^l u_k\right|$.

Consider inverse problem of determining $\left\{a_k(x'), u_k(x), k = \overline{1, m}\right\}$

$$\Delta u_k + a(x') u_k = f_k(u), x \in B, \tag{1}$$

$$u_k(x) = \phi_k(x), x \in \partial B, \tag{2}$$

$$\int_{\gamma_1(x')}^{\gamma_2(x')} u_k(x', x_u)\, dx_n = h_k(x'), x' \in \bar{B}' \tag{3}$$

Here $f_k(p), \phi_k(x), h_k(x')$are given functions.

Considering problems (1)-(3) we make the following assumptions.

1^0.Functions $f_k(p) = f_k(p_1, ..., p_m)$ are continuous for $p \in R^m$; there is such constant $c_0 > 0$ that, for all $p, q \in R^m$

$$|f_k(p) - f_k(q)| \leq c_0 \sum_{k=1}^m |p_k - q_k|;$$

$$2^0.\phi_k(x) \in C^{2+\alpha}(\partial B);$$

$$3^0.h_k(x') \in C^{2+\alpha}\left(\bar{B}'\right);$$

$$4^0.\gamma_1(x'), \gamma_2(x') \in C^{1+\alpha}\left(\bar{B}'\right).$$

Definition 1. *Pair of functions* $\left\{a_k(x'), u_k(x), k = \overline{1, m}\right\}$*called solution of the problem (1)-(3), if*

1. $a_k\left(x'\right)\in C^{\alpha}\left(\bar{B}'\right)$,
2. $u_k\left(x\right)\in C^{2+\alpha}\left(\bar{B}\right)$,
3. *For these functions problem (1)-(3)satisfies in the usual sense.*

The uniqueness theorem, as well as the assessment of the "conditional" stability of the solution of the inverse problems, plays a central role in the study of the questions of their correctness according to Tikhonov.

Let $\left\{{a_k}^i\left(x'\right),{u_k}^i\left(x\right),k=\overline{1,m}\right\}$- be the solution of the problem (1)-(3) with corresponding data

$$g_k^i\left(u^i\right),\phi_k^i\left(x\right),h_k^i\left(x'\right),i=1,2...$$

Theorem. *Let*

1. *Functions* $f_k^i\left(u\right),\phi_k^i\left(x\right),h_k^i\left(x'\right),k=\overline{1,m},i=1,2$ *satisfy conditions* 1^0-3^0, *respectively;*
2. *Exists solution* $\left\{{a_k}^i\left(x'\right),{u_k}^i\left(x\right),k=\overline{1,m},i=1,2\right\}$ *of problem (1)-(3) in the sense of Definition 1*

and they belong to the set $K_\alpha=\left\{(a,u)\mid a\left(x'\right)\in C^{\alpha}\left(\bar{B}'\right),u\left(x\right)\in C^{2+\alpha}\left(\bar{B}\right),\right.$ $\left.a\left(x'\right)\mid\leq c_1,x'\in\bar{B}',\left|D^l{}_x u\left(x\right)\right|\leq c_2,l=0,1,2,x\in\bar{B},c_1,c_2\right\}$ *are some positive numbers.*

Then the solution of the problem (1)-(3) for $x\in\bar{B}$ *is unique and satisfy following stability:*

$$\left\|a^1-a^2\right\|_0+\left\|u^1-u^2\right\|_0\leq c_3\left[\left\|f^1-f^2\right\|_0+\left\|\varphi^1-\varphi^2\right\|_2+\left\|h^1-h^2\right\|_2\right],$$

where $c_3>0$,*-depends on (1)-(3) and the set* K_α.

Inverse sectral problems for one-dimensional Schrödinger operators with additional growing potentials

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The report considers the question of the possibility of studying inverse spectral problems for one-dimensional Schrödinger operators with unbounded potentials. Consider the one-dimensional Schrödinger equation of the form

$$-y'' + Q(x)\,y + q(x)\,y = \lambda y,\ -\infty < x < +\infty,$$

where $Q(x)$ is a growing real potential, and $q(x)$ is a rapidly decreasing real potential. It is shown that for some classes of such operators the inverse spectral problems can be studied in detail by the method of transformation operators. Inverse scattering problems are studied for the Schrödinger equations on an axis with potentials growing indefinitely at the left end and disappearing at the right end. In particular, the inverse scattering problem for the equation

$$-y'' + e^{2x}y + q(x)\,y = \lambda y,\ -\infty < x < +\infty$$

is studied, where the real $q(x)$ potential satisfies the condition

$$\int_{-\infty}^{+\infty} \left(1 + x^2\right) e^x \left|q(x)\right| dx < \infty.$$

Main integral equations of the Gelfand-Levitan-Marchenko type are obtained. The unique solvability of the main equations in the corresponding spaces is established. An efficient algorithm for restoring the perturbation potential from known spectral data is indicated.

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Approximations by the generalized Abel–Poisson integrals on the Weyl–Nagy classes

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Let L be a space of 2π-periodic summable functions and

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

be the Fourier series of $f \in L$.

Further, let C be a subset of the continuous functions from L with the uniform norm $\|f\|_C = \max\limits_t |f(t)|$; L_∞ be a subset of the functions $f \in L$ with the finite norm $\|f\|_\infty = \operatorname{ess\,sup}\limits_t |f(t)|$.

Let $f \in L$, $r > 0$ and β be a real number. If the series

$$\sum_{k=1}^{\infty} k^r \left(a_k \cos\left(kx + \frac{\beta\pi}{2}\right) + b_k \sin\left(kx + \frac{\beta\pi}{2}\right)\right)$$

is the Fourier series of a summable function, then it is denoted by f_β^r and is called the (r,β)-derivative of the function f in the Weyl–Nagy sense (see, e.g., [1]). Let $W_{\beta,\infty}^r$ be the classes of the functions f for which $\left\|f_\beta^r(\cdot)\right\|_\infty \leq 1$.

For 2π-periodic summable on the period function f, by

$$P_\gamma(\delta; f; x) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x+t)\left\{\frac{1}{2} + \sum_{k=1}^{\infty} e^{-\frac{k^\gamma}{\delta}} \cos kt\right\} dt, \quad \delta > 0,\ 0 < \gamma \leq 2,$$

we denote the generalized Abel–Poisson integral (see, e.g., [2]).

In this paper, we consider the problem of asymptotic behavior as $\delta \to \infty$ of the quantity

$$\mathcal{E}(W_{\beta,\infty}^r; P_\gamma(\delta))_C = \sup_{f \in W_{\beta,\infty}^r} \|f(\cdot) - P_\gamma(\delta, f, \cdot)\|_C.$$

Theorem 1. *Let $r > \gamma$. Then the following asymptotic equality holds as $\delta \to \infty$:*

$$\mathcal{E}\left(W_{\beta,\infty}^r; P_\gamma(\delta)\right)_C = \frac{1}{\delta} \sup_{f \in W_{\beta,\infty}^r} \|f_0^\gamma(\cdot)\|_C + O(\Upsilon(\delta, r, \gamma)),$$

where $f_0^\gamma(x)$ is (r,β)-derivative in the Weyl–Nagy sense as $r = \gamma$, $\beta = 0$ and

$$\Upsilon(\delta, r, \gamma) = \begin{cases} \frac{1}{(\sqrt[\gamma]{\delta})^r}, & \gamma < r < 2\gamma, \\ \frac{\ln\delta}{\delta^2}, & r = 2\gamma, \\ \frac{1}{\delta^2}, & r > 2\gamma. \end{cases}$$

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Coulomb control of mechanical linkages

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We describe a method of controlling planar configurations of a mechanical linkage by point charges placed at its vertices. Our approach is analogous to the Coulomb control scenario for systems of nano-particles and yields an algorithm of robust control for polygonal linkages and spider robots. The general setting given below will be illustrated by the concrete results available for quadrilateral linkages and tripod spider robots which will be presented in the talk. Several aspects of our approach have been developed in a series of joint papers with G.Panina (Saint-Petersburg) and D.Siersma (Utrecht). Another application of this approach to the theory of electrostatic ion traps was given in a recent joint paper with G.Giorgadze "Triangles and electrostatic ion traps" (J. Math. Physics, vol.62, 2021).

Recall that a mechanical linkage is defined by a finite weighted graph with positive weights interpreted as Euclidean distances of rigid segments connecting its appropriate vertices. The set of planar configurations of linkage L is denoted by $M(L)$. If a graph consists of one cycle this definition yields a polygonal linkage for which $M(L)$ is simply the set of all polygons with fixed lengths of sides. It is known that, for a generic weighted graph, the set $M(L)$ endowed with a natural topology has a structure of differentiable manifold so we can consider differentiable functions on $M(L)$, one of which is defined below and plays a key role in our setting.

For any configuration P of linkage L with n vertices $(v_1, \dots, v_n)$ and any n-tuple of non-zero real numbers $Q = (q_1, \dots, q_n)$, we denote by $E(Q, P)$ the Coulomb (electrostatic) energy of the system of point charges (Q, P) in which charge q_i is placed at vertex v_i. This yields a differentiable function E on an open subset $G(L)$ of $M(L)$ consisting of the so-called non-degenerate configurations without coinciding vertices. For each non-degenerate planar configuration P_0 of linkage L, we say that n-tuple of non-zero real numbers Q is stationary for P_0 if $E(Q, P)$ considered as function on $G(L)$ has a critical point at configuration P_0. From these definitions follows that the set $S(P_0)$ of stationary charges for P_0 consists of non-zero real solutions to a system of quadratic equations. For certain classes of mechanical linkages including polygonal linkages and spider robots, it can be proven that stationary charges exist for any configuration in $G(L)$. In many cases, there exists a collection of positive stationary charges for P_0 and then configuration P_0 is a point of non-degenerate minimum of Coulomb energy E. Assuming that the shape of L is governed only by electrostatic forces so that its equilibrium configuration always renders a minimum to the Coulomb energy $E(Q, P)$, one concludes that configurations of L can be controlled (navigated) by properly changing the values of stationary charges. These constructions and results obviously suggest several natural problems in the spirit of optimal control theory, which can be solved for various special classes of mechanical linkages. As an illustration, solutions of several arising problems will be presented in some detail for quadrilateral linkages and tripod spider robots.

Algorithms: history and some of their applications

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From ancient times through the Middle Ages, algorithms were developed for a variety of purposes, including multiplication and the solution of quadratic equations. In the Renaissance, algebra and other branches of mathematics began to be formalized, leading to the development of more sophisticated algorithms for a wider range of applications. In the modern era, algorithms have been used for everything from the solution of complex mathematical problems to the analysis of vast amounts of data. In this presentation, we will trace the history of algorithms from their earliest known origins to the present day, focusing on their applications in multiplication and in the solution of the quadratic equations.

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Functors and some topological properties

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We review recent results on interactions between certain (normal and semi-normal) functors and some properties, in particular, some cardinal properties, of topological spaces. Especially, we are interested in the problem of preservation of these properties under influence of a functor.

It is a joint work with F.G. Mukhamadiev and A.K. Sadullaev.

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Bessel property for the system of root vector-functions of the second order differential operator with summable coefficients

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We consider the differential operator

$$L\psi = \psi^{''} + P_1\left(x\right)\psi^{'} + P_2(x)\psi$$

in the interval $G = (0,1)$, with summable matrix coefficients $P_l\left(x\right) = \left(p_{lij}\left(x\right)\right)_{i,j=1}^{m}$, $p_{lij}\left(x\right) \in L_1\left(G\right)$, $l = 1,2;\ m \in N$.

The root vector-functions of the operator L are understood in the generalized interpretation (with respect to boundary condition) [1].

We consider an arbitrary system $\left\{\psi_k\left(x\right)\right\}_{k=1}^{\infty}$ consisting of eigen and associated vector-functions (root vector-functions) of the operator of L. This means that each element of the system $\left\{\psi_k\left(x\right)\right\}_{k=1}^{\infty}$ is identically non-zero, is absolutely continuous together with its first order derivative on $\overline{G}$ and almost everywhere in G satisfies the equation

$$L\psi_k + \lambda\psi_k = \theta_k\psi_{k-1},$$

where θ_k equals either 0 (in this case $\psi_k\left(x\right)$ is an eigen vector-functions), or 1 (in this case $\psi_k\left(x\right)$ is an associated vector-functions, $\lambda_{k-1} = \lambda_k$), $\theta_1 = 1$.

The highest order of root vector-function corresponding to the given eigen vector-function will be called the rank of this eigen vector-function.

Defination. *The system vector-functions* $\left\{\upsilon_k\left(x\right)\right\}_{k=1}^{\infty} \subset L_2^m\left(G\right)$ *is called Bessel (or satisfies the Bessel inequality) if there exist a constant* M *such that for each vector-function* $f\left(x\right) \in L_2^m\left(G\right)$ *the following inequality is fulfilled:*

$$\left(\sum_{k=1}^{\infty}\left|\left(f,\upsilon_k\right)\right|^2\right)^{1/2} \leq M\left\|f\right\|_{2,m}.$$

In the present work we study the Bessel property for the system of root vector-functions of the operator L and prove the following theorem.

Theorem. *Let* $p_{lij}\left(x\right) \in L_1\left(G\right);\ i,j = \overline{1,m};\ l = 1,2;$ *the rank of eigen vector-functions be uniformly bounded and exists a constant* C *such that*

$$\left|Im\mu_k\right| \leq C, \quad k = 1,2,\ldots. \tag{1}$$

Then for the Bessel property of the system $\left\{ \psi_k(x) \, \|\psi_k\|_{2,m}^{-1} \right\}_{k=1}^{\infty}$ *in* $L_2^m(G)$ *it is necessary and sufficient the existence of a constant* M_1 *such that*

$$\sum_{\tau \leq Re\mu_k \leq \tau+1} 1 \leq M_1, \quad \forall \tau \geq 0, \tag{2}$$

where $\mu_k = \sqrt{\lambda_k}$, $Re\mu_k \geq 0$.

Remark. In the sufficient part of the Theorem, the condition of uniform boundedness of the rank of eigen vector-functions as knowingly fulfilled, because it is a corollary of inequality (2).

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Riesz property criterion for the system of root function of second order differential operator

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On the interval $G = (0, 1)$, consider the operator

$$Lu = u'' + q_1(x)\ u' + q_2(x)u$$

with summable coefficients $q_i(x) \in L_1(G)$, $l = 1, 2$.

The root functions (i.e. eigenfunctions and associated function) of the operator L are understood in the generalized sense (irrespective of the boundary condition) [1].

Let $\{u_k(x)\}_{k=1}^{\infty}$ be an arbitrary system of root functions of the operator L, and let $\{\lambda_k\}_{k=1}^{\infty}$be the corresponding system of eigenvalues. Moreover, we assume that any function $u_k(x)$occurs in the system$\{u_k(x)\}_{k=1}^{\infty}$ together with associated functions of lower order.

Definition. *A* system $\{\varphi_k(x)\}_{k=1}^{\infty} \subset L_q(G)$ is called a Riesez system if there exist a constant $M = M(p)$ such that for each function $f(x) \in L_q(G)$, $1 < p \leq 2$, $q = \frac{p}{(p-1)}$, one has the inequality $\left(\sum_{k=1}^{\infty} |(f, \varphi_k)|^q\right)^{\frac{1}{q}} \leq M \|f\|_p$.

Denote $\mu_k = \sqrt{\lambda_k}$, $Re\mu_k \geq 0$.

In the present work, we prove the following result.

Theorem (Riesz property criterion). Let $q_i(x) \in L_1(G)$, $l = 1, 2$, let the lengths of chains of root functions be uniformly bounded, and assume that there exists a constant C_0 such that

$$|Im\mu_k| \leq C_0, \quad k = 1, 2, \ldots . \tag{1}$$

Then a necessary and sufficient condition for the Riesz property of the system $\left\{u_k(x)\ \|u_k\|_q^{-1}\right\}_{k=1}^{\infty}$ in $L_p(G)$ is that there exist a constant M_1 such that

$$\sum_{\tau \leq Re\mu_k \leq \tau+1} 1 \leq M_1, \quad \forall \tau \geq 0. \tag{2}$$

Remark. In the sufficient part of the Theorem, the condition of uniform boundedness of the length of chains of root functions holds automatically as a consequence of inequality (2).

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Upper bound for the approxımatıon error of rbf neural networks wıth two hıdden nodes

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Let $f(x)$ be a given continuous function on some compact subset Q of $\mathbb{R}^d$ and $g(x)$ be any continuous function on $\mathbb{R}$. Consider the approximation of f from the following set of RBF neural networks

$$\mathcal{G}=\mathcal{G}(g,\mathbf{c_1},\mathbf{c_2})=\left\{w_1 g\left(\frac{\|\mathbf{x}-\mathbf{c_1}\|}{\sigma_1}\right)+w_2 g\left(\frac{\|\mathbf{x}-\mathbf{c_2}\|}{\sigma_2}\right): w_i,\sigma_i\in\mathbb{R}, i=1,2\right\}. \quad (1)$$

Note that in (1) $\mathbf{c}_1$ and $\mathbf{c}_2$ are fixed, whileas the numbers w_1, w_2, σ_1, σ_2 variate.
The approximation error is defined as follows

$$E\left(f\right)=E(f,\mathcal{G})\overset{def}{=}\inf_{u\in\mathcal{G}}\left\|f-u\right\|, \left\|f-u\right\|=\max_{x\in\mathcal{Q}}|f(x)-u(x)|.$$

Suppose Q is a compact set in $\mathbb{R}^d$ and $\mathbf{c}_1,\mathbf{c}_2\in\mathbb{R}^d$ are fixed points.

Definition 1. ([1]) *A finite or infinite ordered set $p=(\mathbf{p}_1,\mathbf{p}_2,...)\subset Q$ with $\mathbf{p}_i\neq\mathbf{p}_{i+1}$, and either $\left\|\mathbf{p}_1-\mathbf{c}_1\right\|=\left\|\mathbf{p}_2-\mathbf{c}_1\right\|, \left\|\mathbf{p}_2-\mathbf{c}_2\right\|=\left\|\mathbf{p}_3-\mathbf{c}_2\right\|, \left\|\mathbf{p}_3-\mathbf{c}_1\right\|$ $=\left\|\mathbf{p}_4-\mathbf{c}_1\right\|,...$ or $\left\|\mathbf{p}_1-\mathbf{c}_2\right\|=\left\|\mathbf{p}_2-\mathbf{c}_2\right\|, \left\|\mathbf{p}_2-\mathbf{c}_1\right\|=\left\|\mathbf{p}_3-\mathbf{c}_1\right\|, \left\|\mathbf{p}_3-\mathbf{c}_2\right\|$ $=\left\|\mathbf{p}_4-\mathbf{c}_2\right\|,...$is called a path with respect to the centers $\mathbf{c}_1$ and $\mathbf{c}_2$.*

We use the term “path” instead of the long expression “path with respect to the centers $\mathbf{c}_1$ and $\mathbf{c}_2$”. A finite path $(\mathbf{p}_1,\mathbf{p}_2,...,\mathbf{p}_{2n})$ is said to be closed if $(\mathbf{p}_1,\mathbf{p}_2,...,\mathbf{p}_{2n},\mathbf{p}_1)$ is also a path.

Consider also a class of function $\mathcal{D}$.

$$\mathcal{D}=\left\{r_1(\|\mathbf{x}-\mathbf{c}_1\|)+r_2(\|\mathbf{x}-\mathbf{c}_2\|):\ r_i\in C(\mathbb{R}),\ i=1,2\right\}.$$

We associate a closed path $p=(\mathbf{p}_1,\mathbf{p}_2,...,\mathbf{p}_{2n})$ with the functional

$$G_p(f)=\frac{1}{2n}\sum_{k=1}^{2n}(-1)^{k+1}f(\mathbf{p}_k).$$

This functional has the following obvious properties:
(a) If $r\in\mathcal{D}$, then $G_p(r)=0$.
(b) $\|G_p\|\leq 1$ and if $p_i\neq p_j$ for all $i\neq j$, $1\leq i,j\leq 2n$, then $\|G_p\|=1$. [2, 3]

Theorem 1. *Assume $Q\subset\mathbb{R}^d$ be a compact set and $f\in C(Q)$. Suppose the following conditions hold.*

1) f has a best approximation in $\mathcal{G}=\mathcal{G}(g,\mathbf{c_1},\mathbf{c_2})$;

2) there exists a positive integer N *such that any path* $(\mathbf{p}_1, ..., \mathbf{p}_n) \subset Q$, $n > N$, *or a subpath of it can be made closed by adding not more than* N *points of* Q.

Then for any activation function $g(\|x\|)$, $g : \mathbb{R} \to \mathbb{R}$ *is integrable, continuous and satisfies* $\int\limits_0^\infty g(t)dt \neq 0$ *([4]), the approximation error by RBF neural networks with two hidden nodes* $\mathcal{G} = \mathcal{G}(g, \mathbf{c_1}, \mathbf{c_2})$ *satisfy the following inequality:*

$$E\left(f, \mathcal{G}(g)\right) \leq \sup_{p \subset Q} \left|G_p(f)\right|,$$

where the sup is taken over all closed paths.

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Optimization of hyperbolic-type polyhedral differential inclusions

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The paper concerns the optimization of the Lagrange problem with differential inclusions (DFIs) of the hyperbolic type given by polyhedral set-valued mappings. To do this, the corresponding discrete problem with a polyhedral discrete inclusion is defined. Next, using the Farkas theorem, locally conjugate mappings are calculated and necessary, and sufficient optimality conditions for polyhedral hyperbolic discrete inclusions are proven. Thus, using only data on the polyhedral nature of the problem, and the discretization method for a hyperbolic type polyhedral discrete-approximate problem the necessary and sufficient conditions of optimality, and then by passing to the limit in the form of the Euler-Lagrange type adjoint DFI, sufficient optimality conditions for a continuous problem are formulated.

It is well known that in the field of optimal control theory the described DFIs with ordinary and partial DFIs [1, 2, 3, 4] have made great progress. A set-valued mapping $F^*(\cdot;(u,v)) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by $F^*(v^*;(u,v)) = \{u^* : (u^*, -v^*) \in K_F^*(u,v)\}$ is called the locally adjoint mapping (LAM) to F at a point (u,v) where $K_F^*(u,v) \equiv K_{\mathrm{gph}\, F}^*(u,v)$ is the dual cone to the cone $K_F(u,v)$.

First, consider the problem with polyhedral discrete inclusions of hyperbolic type, which is the basis for further research of the presentation labeled by (HPD):

$$\text{minimize} \sum_{(x,t)\in \mathrm{L}\times\mathrm{T}} g\left(u_{x,t}, x, t\right),$$

$$\text{(HPD)} \quad u_{x,t+1} \in F\left(u_{x,t-1}, u_{x-1,t}, u_{x,t}, u_{x+1,t}\right), u_{0,t} = \alpha_{0t}, u_{L,t} = \alpha_{Lt}, u_{x,0} = \beta_{x0},$$

$$u_{x,T} = \beta_{xT}, \quad F\left(u_{x,t-1}, u_{x-1,t}, u_{x,t}, u_{x+1,t}\right) = \{u_{x,t+1} : A_0 u_{x,t-1} + A_1 u_{x-1,t} + A_2 u_{x,t} + A_3 u_{x+1,t} - B u_{x,t+1} \le d\},$$
$$\mathrm{T} = \{t : t = 1, \dots, T-1\}, \mathrm{L} = \{x : x = 1, \dots, L-1\},$$

where L and T are fixed natural numbers, $g(\cdot, x, t) : \mathbb{R}^n \to \mathbb{R}^1 \cup \{+\infty\}, F : \mathbb{R}^{4n} \rightrightarrows \mathbb{R}^n$ is a polyhedral set-valued mapping, α_{0t}, α_{Lt} and β_{x0}, β_{xT} are fixed vectors for all x and t, respectively, $A_i (i = 0,1,2,3), B$ are $s \times n$ matrices, d is an s-dimensional vector-colomn. A set of vectors $\{u_{x,t}\}_D = \{u_{x,t} : (x,t) \in D\}$, where $D = \{(x,t) : x = 0, ..L; t = 0, \dots, T, (x,t) \neq$

$(0,0)$, $(L,0)$, $(0,T)$, $(L,T)\}$ is called a feasible solution for the problem (HPD).
The second part of the paper is devoted to the study of a polyhedral problem with hyperbolic DFIs in one spatial dimension:

$$\text{minimize } J[u(\cdot,\cdot)] = \iint_Q g(u(x,t),x,t)dxdt,$$
$$\text{(HPC)}\quad \tfrac{\partial^2 u(x,t)}{\partial t^2} - \tfrac{\partial^2 u(x,t)}{\partial x^2} \in G(u(x,t)), \quad (x,t)\in Q=[0,L]\times[0,T]$$
$$u(x,0)=\alpha_0(x), \tfrac{\partial u(x,0)}{\partial t}=\alpha_1(x), u(0,t)=\beta_0, u(L,t)=\beta_L(t),\ G(u(x,t))=$$
$$\left\{\tfrac{\partial^2 u(x,t)}{\partial t^2} - \tfrac{\partial^2 u(x,t)}{\partial x^2} : Pu(x,t) - K\left(\tfrac{\partial^2 u(x,t)}{\partial t^2} - \tfrac{\partial^2 u(x,t)}{\partial x^2}\right) \le \omega\right\},$$

where $G:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a polyhedral set-valued mapping, and $\alpha_0(x), \alpha_1(x)$ and $\beta_0(t), \beta_{\mathrm{L}}(t)$ are given continuous functions, P, K are $s\times n$ matrices, ω is an s-dimensional vector-column, For convenience, we assume throughout the context that feasible solutions are classical solutions. To prove Theorem 1 of the problem (HPD), we have used Theorems 6.55 of Mahmudov's monograph [4, p.350] concerning the optimality of discrete inclusions in the general setting.
Theorem 1 *For optimality of the solution $\{\tilde u_{x,t}\}_D$ in the optimization problem (HPD), it is necessary and sufficient that there exist vectors $\lambda_{x,t}\ge 0$, not all zero such that*
(i) $B^*\lambda_{x,t-1} - A_0^*\lambda_{x,t+1} - A_1^*\lambda_{x+1,t} - A_2^*\lambda_{x,t} - A_3^*\lambda_{x-1,t} \in \partial g(\tilde u_{x,t},x,t) \quad \lambda_{x,t}\ge 0,$
$\langle A_0\tilde u_{x,t-1} + A_1\tilde u_{x-1,t} + A_2\tilde u_{x,t} + A_3\tilde u_{x+1,t} - B\tilde u_{x,t+1} - d, \lambda_{x,t}\rangle = 0,$
(ii) $A_0^*\lambda_{x,T}=0, B^*\lambda_{x,T-1}=0, A_3^*\lambda_{0,t}=0, A_1^*\lambda_{L,t}=0, (x,t)\in \mathrm{T}\times\mathrm{L}.$
Let us introduce the following difference operators $\Omega_{1\delta}=\Omega_1$ and $\Omega_{2h}=\Omega_2$, defined on the two-point models [4] :

$$\Omega_1 u(x,t) = \frac{u(x+\delta,t)-2u(x,t)+u(x-\delta,t)}{\delta^2};$$
$$\Omega_2 u(x,t) = \frac{u(x,t+h)-2u(x,t)+u(x,t-h)}{h^2},$$
$$x\in \mathrm{N}=\{\delta,\dots,L-\delta\}; t\in \Xi=\{h,\dots,T-h\}.$$

Theorem 2. *For the optimality of the solution $\{\tilde u(x,t),(x,t)\in \mathrm{N}\times\Xi\}$ in the optimization problem with hyperbolic polyhedral discrete-approximate inclusions, it is necessary and sufficient that there exist nonnegative vectors $\{\lambda(x,t)\ge 0,(x,t)\in N\times\Xi\}$, not all zero such that:*
(i) $-P^*\lambda(x,t) + K^*(\Omega_2\lambda(x,t) - \Omega_1\lambda(x,t)) \in \partial g(\tilde u_{x,t},x,t), \lambda(x,t)\ge 0,$
$\langle P\tilde u(x,t) - K(\Omega_2\tilde u(x,t) - \Omega_1\tilde u(x,t)) - \omega, \lambda(x,t)\rangle = 0, (x,t)\in \mathrm{N}\times\Xi,$
(ii) $K^*\lambda(x,T)=0, K^*\lambda(x,T-h)=0, x\in\mathrm{N}, K^*\lambda(0,t)=0, K^*\lambda(L,t)=0,$
$t\in\Xi.$
Theorem 3 *For the optimality of the solution $\tilde u(x,t)$ in the problem (HPC) it is sufficient that there exists a nonnegative classical solutions $\lambda(x,t)\ge 0$, $(x,t)\in Q$, satisfying the EulerLagrange type polyhedral DFIs (i) and boundary-value conditions (ii):*
(i) $-P^*\lambda(x,t) + K^*\left(\frac{\partial^2\lambda(x,t)}{\partial t^2} - \frac{\partial^2\lambda(x,t)}{\partial x^2}\right) \in \partial g(\tilde u_{x,t},x,t), \lambda(x,t)\ge 0,$

$$\left\langle P\tilde{u}(x,t) - K\left(\frac{\partial^2\tilde{u}(x,t)}{\partial t^2} - \frac{\partial^2\tilde{u}(x,t)}{\partial x^2}\right) - \omega, \lambda(x,t)\right\rangle = 0,\ Q = [0,L]\times[0,T]$$

(ii) $K^*\lambda(x,T) = 0,\ K^*\lambda(0,t) = 0,\ K^*\lambda(L,t) = 0,\ K^*\frac{\partial\lambda(x,T)}{\partial t} = 0.$

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On the global solvability of the Cauchy problem for an infinite-dimensional system of nonlinear differential equations

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We consider the following infinite-dimensional system of nonlinear evolution equations:

$$\begin{aligned} &\dot{c}_n = c_n \left(\alpha \left(c_{n+1} - c_{n-1} \right) - \beta \left(\left(c_{n+1} - c_{n-1} \right) \sum_{k=0}^{2} c_{n+k} \right) \right), \\ &c_n = c_n\left(t\right), \, n \geq 0, t \in (0, \infty] \,, \; c_{-1} = 0, \; \cdot = \tfrac{d}{dt}, \end{aligned} \tag{1}$$

where α, β are real numbers. This system was first studied in [1], where it was also found there that system of equations (1) can be integrated using the method of the inverse spectral problem. With $\alpha = 1$, $\beta = 0$ system of equations (1) represents the well-known Volterra model, which was studied by the method of the inverse spectral problem by many authors [1-4].

For the system of equations (1), we formulate the Cauchy problem: find the solution of this system, $c\left(t\right) = \left(c_n(t)\right)_{n \geq 0}$, from the initial condition

$$c_n(0) = \hat{c}_n > 0, \; n \geq 0 \tag{2}$$

where the sequence $\hat{c}_n$ satisfies the condition $\sum_{n>0} |n| \left|\hat{c}_n - 1\right| < \infty$. We will seek the solution $c\left(t\right) = \left(c_n(t)\right)_{n \geq 0}$ of the problem (1) and (2) such that $x_n\left(t\right) = c_n\left(t\right) - 1$ is a rapidly decreasing function, i.e., a function satisfying for any $T > 0$ the inequality

$$\left\| M_1\left(t\right) \right\|_{C[0,T]} < \infty, \tag{3}$$

where $M_1\left(t\right) = \sum_{n>0} \left(1 + |n|\right) \left|x_n\left(t\right)\right|$.

In this work, the global solvability of problems (1) and (2) in class (3) is established.

Theorem. *There exists a unique solution of problem* (1) *and* (2) *in class* (3), *if* $M_1\left(0\right) < \infty$.

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The Cauchy problem for a differential equation with the generalized Hilfer operator of fractional order

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In this research work, it is proved that a more general initial value problem for a differential equation involving the generalization of the Hilfer operator has a unique solution. We note that the generalization of fractional order operators in the sense of Riemann-Liouville and Caputo plays an important role in fractional order analysis [1].

We investigate the problem of finding a solution of the equation

$$D_{0^+}^{(\alpha,\beta)\mu}u(t)+\lambda u(t)=f(t), \tag{1}$$

which satisfies the initial condition

$$\lim_{t\to 0^+} t^{1-\gamma-\xi} I_{0^+}^{\xi} u(t)=A, \tag{2}$$

where α, β, ξ, μ, λ, A are real numbers such that $0<\alpha$, $\beta\le 1$, $\mu\in[0,1]$, $0\le\xi<1$, and $f(t)$ is a given function.

$$D_{0^+}^{(\alpha,\beta)\mu}u(t)=I_{0^+}^{\mu(1-\alpha)}\frac{d}{dt}I_{0^+}^{(1-\mu)(1-\beta)}u(t)$$

is the generalized Hilfer operator of the type μ of fractional order α, β [2],

$$I_{0^+}^{\eta}f(t)=\frac{1}{\Gamma(\eta)}\int_0^t (t-s)^{\eta-1}f(s)ds$$

is the Riemann-Liouville integral operator of fractional order α [3].

The following assertion is valid:

Theorem 1. *If $1-\gamma-\xi\ge 0$ and $f(t)\in C_{1-\gamma}[0,\infty)$, then there exists the unique solution $u(t)\in C_{1-\gamma}[0,\infty)$ of the problem (1)-(2) and this solution has the form*

$$u(t)=A\Gamma(\gamma+\xi)t^{\gamma-1}E_{\delta,\gamma}\left(-\lambda t^{\delta}\right)+\int_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(-\lambda (t-s)^{\delta}\right)f(s)ds. \tag{3}$$

Here $\gamma=\beta+\mu(1-\beta)$, $\delta=\beta+\mu(\alpha-\beta)$, and

$$E_{\delta,\gamma}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\delta n+\gamma)}$$

is the Mittag-Leffler function of two parameters [3], and the class $C_\nu[0,\infty)$ of functions is defined as follows:

$$C_\nu[0,\infty) = \{g(t): \|g\|_{C_\nu} = \|t^\nu g(t)\|_C < \infty\}.$$

Proof. The general solution of (1) is determined in the form [2]

$$u(t) = Ct^{\gamma-1}E_{\delta,\gamma}\left(-\lambda t^\delta\right) + \int\limits_0^t (t-s)^{\delta-1}E_{\delta,\delta}\left(-\lambda(t-s)^\delta\right) f(s)ds, \tag{4}$$

where C is an arbitrary constant number.

First of all, we calculate $I_{0+}^{\xi}u(t)$, i.e.

$$I_{0+}^{\xi}u(t) = Ct^{\gamma+\xi-1}E_{\delta,\gamma+\xi}\left(-\lambda t^\delta\right) + \int\limits_0^t (t-z)^{\delta+\xi-1}E_{\delta,\delta}\left(-\lambda(t-z)^\delta\right) f(z)dz. \tag{5}$$

Here, for obtaining (5) we have used the formula [4]

$$\frac{1}{\Gamma(\nu)}\int\limits_0^z (z-s)^{\nu-1}s^{\beta-1}E_{\alpha,\beta}\left(\lambda s^\alpha\right) ds = z^{\beta+\nu-1}E_{\alpha,\beta+\nu}(\lambda z^\alpha).$$

Substituting (5) into (2) and considering $1-\gamma-\xi \geq 0$ we find

$$C = A\Gamma(\gamma+\xi).$$

On the basis of (4), it follows that the solution of the considered problem has the form (3). If the given function $f(t)$ satisfies the condition $f(t) \in C_{1-\gamma}[0,\infty)$, then, according to (3), it is not difficult to show $u(t) \in C_{1-\gamma}[0,\infty)$.

A similar problem in the Riemann-Liouville case was considered in [5].

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Problems on the curves in a complex plane related with classic approximation theorems

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We consider the following model case of Jackson, Jackson-Bernstein, Bernstein and Nikolsky-Timan-Dzjadyk classic theorems.

Theorem 1. *(Jackson's). Let* $f \in Lip_{[a,\, b]}\alpha,\ (0 < \alpha \leq 1)$*, then*

$$E_n\left(f,\, [a,\, b]\,\right) \leq \frac{const}{n}.$$

Theorem 2. *(Jackson-Bernstein). In order that*

$$f \in Lip_{[0,\, 2\pi]}\alpha,\, (0 < \alpha < 1) \Leftrightarrow E_n\left(f;\, [0,\, 2\pi]\,\right) \leq \frac{const}{n^{\alpha}}.$$

Theorem 3. *(Bernstein). Let* $f_0\left(\theta\right) = f\left(\cos\theta\right) = f(x),\ x = [-1, 1]$.
In order that

$$f_0\left(\theta\right) \in Lip_{[0,\, 2\pi]}\alpha,\ (0 < \alpha < 1) \Leftrightarrow E_n\left(f;\, [-1,\, 1]\,\right) \leq \frac{const}{n^{\alpha}}.$$

Theorem 4. *(Nikolsky-Timan-Dzjadyk). In order that*

$$f \in Lip_{[-1,1]}\alpha \quad (0 < \alpha < 1) \Leftrightarrow \exists P_n$$

for which $\forall x \in [-1,\, 1]$

$$\left|f\left(x\right) - P_n\left(x\right)\right| \leq const\left(\frac{\sqrt{1 - x^2}}{n} + \frac{|x|}{n^2}\right)^{\alpha}.$$

The problems we are interested may be formulated in a general form: What necessary and sufficient conditions should (must) satisfy a class of curves in a complex plane, in order appropriate classic theorem of approximation be fulfilled (true) on it.

These problems are urgent both in the metric C and in the metric L_p . Notice that the problem related with Jackson-Bernstein theorem was formulated by J.Walsh and the problem related with Jackson theorem by D. Newman.

Some results on the global behavior for the solution of a nonlinear fourth order equation with nonlinear boundary conditions

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We consider the following problem

$$U_{tt} + \Delta^2 u - \beta\Delta u_t + \gamma u_t + f(u) = 0, \quad (x,t) \in \Omega \times [0,T] \tag{1}$$

$$u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \quad x \in \Omega \tag{2}$$

$$\frac{\partial u}{\partial n} = 0, \quad (x,t) \in \partial\Omega \times [0,T] \tag{3}$$

$$\frac{\partial \Delta u}{\partial n} = g(u), \quad (x,t) \in \partial\Omega \times [0,T] \tag{4}$$

where $\Omega \in \mathbb{R}^n$ is bounded domain with smooth boundary $\partial\Omega$, $f(u)$ and $g(u)$ are some nonlinear functions, β and γ are some positive numbers, $\Delta u = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is Laplace operator, $\frac{\partial}{\partial n}$ is the derivative on the external normal at $\partial\Omega$.

A lot of papers have been devoted to studying the behavior of solutions to equations of type (1) with different boundary conditions, as evidenced by references [1]-[6].

In general, these works address the presence of nonlinearity in the equation.

In this work, we study a question of stabilization and "blow up" of solutions for problems (1)-(4) when the boundary function has some smoothing properties.

For the problem (1)-(4) it is proved following theorems:

Theorem 1. *Let for the functions $f(\tau)$ and $g(s)$ satisfy following conditions*

$$F(u) = \int_0^u f(\tau)d\tau \geq 0, \; f(0) = 0, \quad G(0) = \int_0^u g(s)ds \geq 0, \; f(0) = 0$$

however $uf(u) - F(u) > 0$, $ug(u) - G(u) > 0$ for every $u \in \mathbb{R}$ and besides it $g(0)$ is such function, that $G(u) \geq Mu^2$, $\forall u \in \mathbb{R}$, where M is some positive number. Then for every solution $u(x,t) \in W_2^1(0,T;W_2^2(\Omega)) \cap W_2^2(0,T,L_2(\Omega))$

1) there exists $0 < \eta \leq \frac{3}{\delta}$ such that $\frac{\eta(10-\eta)}{2(4-\eta)} \leq k \leq \frac{2M - C\eta_1 + 2\eta}{2}$, $M \geq \frac{\eta(10-\eta)}{2(4-\eta)} + \frac{\delta}{2}\eta - \eta$ the solution stabilizing in sense of $\|u_t\|_{L_2} + \|u_t\|_{W_2^1(\Omega)} \to 0$, as $t \to \infty$,

2) there exists $0 < \eta \leq \frac{3}{2\delta}$ such that $\frac{\eta(8-\eta)}{2(4-\eta)} \leq k \leq \frac{2M - \eta(2\delta - 1)}{2}$, $M \geq \frac{\eta(8-\eta)}{4-\eta} + \delta\eta + \frac{1}{2}\eta$ the solution stabilizing in sense of $\|u_t\|_{L_2} + \|u_t\|_{L_2(\Omega)} \to 0$, as $t \to \infty$.

Theorem 2. *Let for some $\alpha > 0$ and $\forall u \in \mathbb{R}$ satisfy following conditions*

$$2(2\alpha + 1)f(u) - uf(u) \geq 0$$

$$2(2\alpha+1)G(u)-ug(u)\geq 0$$
$$\frac{1}{2}\|u_1(x)\|^2+\frac{1}{2}\|\Delta u_0(x)\|^2+\int_{\partial\Omega}G(u_0)ds+\int_{\Omega}F(u_0)dx\leq 0$$
$$(u_0,u_1)>0$$

Then for every solution $u(x,t)\in W_2^1(0,T;W_2^2(\Omega))\cap W_2^2(0,T,L_2(\Omega))$ and the problem (1)-(4), there exists $t_0\to\infty$ such that for every $\delta>0$

$$\|u(x_1,t)\|^2+\delta\int_0^t\|\Delta u(x,\tau)\|^2d\tau\to\infty$$

as

$$t\to t_0\leq t_1=\frac{\|u_0\|^2+\delta T\|\Delta u_0\|^2}{2(u_0,u_1)}.$$

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About special subspaces of Marcinkiewicz and Morrey spaces

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Let $X(R^n)$ be some Banach function space, $\Omega \subset R^n$ be an arbitrary domain and $X(\Omega)$ will be corresponding space induced by restriction of all functions from $X(R^n)$ on Ω. For arbitrary function $f \in X(\Omega)$ and for arbitrary $\delta \in R^n$, by we denote the additive shift operator, defined as

$$(T_\delta f)(x) = \begin{cases} f(x+\delta), & x+\delta \in \Omega, \\ 0, & x+\delta \notin \Omega. \end{cases}$$

By $X_s\ (\Omega)$ we denote the subspace of all functions from $X(\Omega)$, which have the following property: $\|T_\delta(f) - f\|_{X(K)} \to 0,\ \delta \to 0.$. Let $X_a(\Omega)$ and $X_b(\Omega)$ are subspaces of all absolutely continuous functions and closure of all simple functions correspondingly. Consider the following spaces.

Marcinkiewicz space $X = M^{p,\lambda}(\Omega)$, $(1 \le p < +\infty,\ 0 < \lambda < n)$. This is a Banach function space of measurable (in Lebesgue sense) functions on Ω with the norm $\|f\|_{p,\lambda} = \sup\limits_I \left(\frac{1}{|I|^{\frac{\lambda}{n}}}\int_I |f|^p\,dt\right)^{\frac{1}{p}}$, where $I \subset R^n$ is an arbitrary measurable subset.

Morrey space $L^{p,\lambda}(\Omega)$, $(1 \le p < \infty,\ 0 < \lambda < n)$**.** The norm in this space is defined as

$$\|f\|_{p,\lambda} = \sup_{B_r \subset R^n}\left(\frac{1}{r^\lambda}\int_{B_r}|f|^p\,dx\right)^{\frac{1}{p}},$$

where sup is got on all balls.

It is well known that in these spaces $X_a(\Omega) = X_b(\Omega) = X_s(\Omega) = \overline{C_0^\infty(\Omega)}$ holds. The subspaces $X_s(\Omega)$ of these spaces we will denote by $M_s^{p,\lambda}(\Omega)$ and $L_s^{p,\lambda}(\Omega)$ correspondingly.

We assert that $M_s^{p,\lambda}(\Omega) \neq L_s^{p,\lambda}(\Omega)$. Consider the case $\Omega = (0;1)$ and the subsets

$$E_n = \bigcup_{k=\overline{1,n}} (a_{nk}; b_{nk}),\ a_{n1} = 0,\ b_{nn} = 1,$$

$$b_{nk} = a_{nk} + x_n,\ a_{n(k+1)} = b_{nk} + y_n,\ n(x_n + y_n) = 1.$$

Let's calculate the norms of characteristic functions χ_{E_n} of these subsets. Taking into account that $M^{p,\lambda}$ is rearrangement-invariant, we have

$$\|\chi_{E_n}\|_{M^{p,\lambda}(0;1)} = \left\|\chi_{(0;nx_n)}\right\|_{M^{p,\lambda}(0;1)} = \left(\frac{1}{(nx_n)^\lambda}\int_0^{nx} dx\right)^{\frac{1}{p}} = (nx_n)^{\frac{1-\lambda}{p}}$$

where we used the relation

$$\frac{1}{a^{\lambda}}\int_0^a dx = a^{1-\lambda} < b^{1-\lambda} = \frac{1}{b^{\lambda}}\int_0^b dx,\ a<b\,.$$

In Morrey space case, we have the following estimates:

$$0<z\le x_n \Rightarrow \tfrac{1}{z^{\lambda}}\int_0^z dx = z^{1-\lambda} \le {x_n}^{1-\lambda} = \tfrac{1}{{x_n}^{\lambda}}\int_0^{x_n} dx,$$

$$\tfrac{1}{(a_{k+1}+z)^{\lambda}}\int_0^{a_{k+1}+z}\chi_{(0;a_k+z)\bigcap E_n}dx = \tfrac{kx_n+z}{(k(x_n+y_n)+z)^{\lambda}} =$$

$$=(kx_n+z)^{1-\lambda}\left(\tfrac{kx_n+z}{kx_n+ky_n+z}\right)^{\lambda} = (kx_n+z)^{1-\lambda}\left(1-\tfrac{ky_n}{kx_n+ky_n+z}\right)^{\lambda}\le$$

$$\le((k+1)x_n)^{1-\lambda}\left(\tfrac{(k+1)x_n}{(k+1)x_n+ky_n}\right)^{\lambda} = \tfrac{(k+1)x_n}{((k+1)x_n+ky_n)^{\lambda}}.$$

Let $y_n \ge t_n x_n : n^{1-\lambda} < {t_n}^{\lambda}$, or $(nt_n)^{\lambda} > n$, $t_n > 2$. Then we have

$$\frac{(k+1)\,x_n}{((k+1)\,x_n+ky_n)^{\lambda}} \le \frac{(k+1)\,x_n}{(t_n\,(k+1)\,x_n)^{\lambda}} \le \frac{n^{1-\lambda}}{{t_n}^{\lambda}}{x_n}^{1-\lambda} < {x_n}^{1-\lambda},$$

from which it follows that $\|\chi_{E_n}\|_{L^{p,\lambda}(0;1)} = {x_n}^{1-\lambda}$. Finally, as a result, we have

$$\frac{\|\chi_{E_n}\|_{M^{p,\lambda}(0;1)}}{\|\chi_{E_n}\|_{L^{p,\lambda}(0;1)}} = n^{\frac{1-\lambda}{p}} \to \infty,\ n\to\infty,$$

i.e. the embedding $L^{p,\lambda}\subset M^{p,\lambda}$ is impossible.

On dirichlet problem for a class of non-uniformly parabolic equations with measure data

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In this note, we relate to the solvability of the first boundary value problem for a class of linear non-uniformly parabolic equation

$$(A)\begin{cases} \frac{\partial u}{\partial t}-\frac{\partial}{\partial z_i}\left(a_{ij}\left(t,z\right)\frac{\partial u}{\partial z_j}\right)=f\left(t,z\right),\ (t,z)\in Q_T, \\ u\left(0,z\right)=g\left(z\right),z\in\Omega, \\ u\Big|_{S_T}=0, \end{cases}$$

where $S_T=\partial\Omega\times(0,T)$ and the coefficients matrix $A=\|a_{ij}\left(t,z\right)\|\,(1\le i,j\le N)$ positively defined a.e $(t,z)\in\mathbb{R}^{N+1}\cap\{t\ge 0\}$ and its elements $a_{ij}(t,z)$, $i,j=1,2,...N$ are measurable functions (and Lipschitz continuous on variable t for almost $z\in\Omega$). The $\Omega\subset\mathbb{R}^N$ is a bounded domain with Lipschitz boundary, $Q=\Omega\times[0,T]$. We assume the non-uniform parabolic condition: there exist positive constants c_1,c_2 such that

$$c_1(\omega(x)\left|\xi\right|^2+\left|\eta\right|^2)\le A(z)\zeta\cdot\zeta\le c_2(\omega(x)\left|\xi\right|^2+\left|\eta\right|^2)$$

a.e. $z\in\Omega$, with $\forall\zeta=(\xi,\eta)\in\mathbb{R}^N, N=n+m,\ \xi\in\mathbb{R}^n,\ \eta\in\mathbb{R}^m; n,m\ge 1.$

We establish the weak solvability of the problem (A) assuming the f and g to be positive Radon measures, the degeneration $\omega(x):\mathbb{R}^n\to[0,\infty]$ is positive and finite a.e. $\mathbb{R}^n$ and is taken from the suitable Muckenhoupt class A_p.

In the proofs, we use the ideas from [1] and the non-uniformly gradient inequalities of Poincare-Sobolev's type [2, 3].

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On determining the time to corrosive failure with nonsteady potential changes

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It is known that metallic structural elements working in aggressive media fail after a certain time. Determination of time to failure is one of the problems in studying the corrosion of materials and products under the joint action of load and aggressive medium. There exist many factors influencing on current material corrosion process among which we can distinguish the followings: temperature and concentration of active components of aggressive medium, corrosion potential, and mechanical stress.

In the paper, we consider the case of a corrosive process when under some temperature and concentration of active components of an aggressive medium, depending on the applied constant stress the corrosion potential changes in time. The damage accumulation concept is used. The corrosion process is determined as a process of continuous accumulation of a certain type of damage; a non-negative function monotonically increasing in time and characterizing the corrosion degree, in other words, the degree of accumulated corrosive damages, is introduced. We obtain a theoretical formula that determines the time to the corrosive failure of metals under mechanical stress with an unsteady potential change in the corrosion process:

$$t_* = t_0\,(\sigma,\, u_0)\,\left(L_1 + L_2 \frac{u_b - u_0}{u_s - u_0}\right). \tag{1}$$

Here t_* is time to the corrosive failure of metals with nonsteady potential change $u : u = u\,(\sigma,\, t)$ where t is time, $\sigma = const$ is mechanical stress; the quantity u_s is some standard potential; the quantities $u_0 = u\ (\sigma,\, 0)$ $u_b = u\ (\sigma,\, t_*)$ for each given stress σ can be measured in experiments; L_1 , L_2 are constants, $t_0 = t_0\ (\sigma,\, u_0)$ is a universal function of the system "metal – corrosive medium", the time to failure of the experimental sample under different constants of the stress σ, of the potential u_0. The technique of experimental determination of the constants L_1 , L_2, and also the dependencies $t_0 = t_0\ (\sigma,\, u_0)\,;\ u_0 = u_0\ (\sigma,\, 0)\,;\ u_b = u\ (\sigma,\, t_*)$ is represented. Some experimental data are processed to verify the well-posedness of formula (1) for determining the time to corrosive failure with the unsteady change of potential. A satisfactory coincidence of calculated and experimental data was obtained.

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Solution of the problem of the long durability of hollow shaft at torsion with the account of its damageability

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Hollow shaft and pipes are one of the most widespread elements of designs, machines, and mechanisms, their basic kind of loading is the transferred torque.

For cases when the material of a hollow shaft possesses cylindrical axis-symmetrical anisotropy, it is accepted, that mechanical modules depend only on the current radius. In the further, the cylindrical system of coordinates r, θ, z is used.

The object of research is the body limited by surfaces of coaxial circular cylinders with radii a and b length, equal to 1. We shall consider one end face fixed motionlessly, and the other is considered under the action of efforts led to the twisted moment of M. For the case of the anisotropic cylinder, its anisotropy is considered cylindrical, and the axis of anisotropy coincides with a geometrical axis and is accepted as axis Z; we shall place the origin of axis Z at a loose end face. The following restrictions are imposed on anisotropy: elastic properties are like that relative radial, ring, and axial deformations do not depend on tangent stresses in a plane of transversal section; the basic modules of elasticity do not depend on the ring and axial coordinate. These conditions can be satisfied, assuming, that among all displacements only ring and axial displacements are distinct from zero, and among all stresses, only tangent stress in a plane of transversal section, perpendicular to radius is distinct from zero. Then from the general formulas of the theory of elasticity of an anisotropic body follows, that only the module of shift G is connected with this pressure $\tau_{\theta z}$. We shall consider it variable along the radius of the transversal section: $G = G(r)$.

Firstly we shall consider a problem of the scattered destruction of an isotropic round hollow shaft in which the transversal section represents a concentric ring with internal radius r and external radius R. The external torque M is assumed to be constant.

The distribution of tangent stress τ within the limits of the hereditary theory of damageability will be as well as for an elastic shaft, namely:

$$\tau = \frac{M}{J_p}\rho \tag{1}$$

where ρ is current radius, and J_ρ is the polar moment of inertia of transversal section:

$$J_\rho = \frac{\pi}{2}\left(R^4 - r^4\right). \tag{2}$$

The greatest stress arises at an external surface $\rho = R$ and is equal:

$$\tau_{\max} = \frac{M}{W_k};\ \ W_k = \frac{\pi r^4}{2R}\left(\left(\frac{R}{r}\right)^4 - 1\right). \tag{3}$$

Here W_k is the moment of resistance at torsion.

As a criterion of destruction, we shall accept:

$$\tau + K^{*}\tau = \tau_0 \tag{4}$$

where τ_0 is an instant strength at shift, and K^{*} is the operator of damageability which at monotonously increasing loading has the same structure, as the usual integrated operator of linear viscous-elasticity:

$$K^{*} \cdot \tau = \int_0^t K\left(t-\xi\right)\tau\left(\xi\right)d\xi \tag{5}$$

where $K\left(t-\xi\right)$ is a kernel of damageability.

The time of primary destruction and the equation of front of destruction turns out from the criterion of destruction (4) with the account of a kind and a level of the stress state.

For a case of cylindrically anisotropic material the power law of change of the module of shift G on radius: $G = g\left(\frac{r}{b}\right)^n$; $n > 0$.

Using the criterion of durability on the greatest stress [1] with the account of uniquely distinct from zero tangent stress we have:

$$\tau_{\theta z}\left(t,t\right) + \int_0^t K\left(t-s\right)\tau_{\theta\, z}\left(t,s\right)ds = \tau_0. \tag{6}$$

As it is known the stress $\tau_{\theta z}$ has various representations for various values of parameter n of nonlinear distribution on the thickness of a pipe of the module of shift [2]. For $n > 0$ it has the form:

$$\tau_{\theta z} = \frac{M\left(4+n\right)}{2\pi}\frac{r^{1+n}}{b^{4+n} - a^{4+n}}. \tag{7}$$

The problem about the origin and development of a zone of destruction in an isotropic and cylindrically anisotropic hollow cylinder is put and solved at torsion when the module of shift is changed on section. Formulas of the incubatory period are deduced. The integrated equations on motion of the front of destruction are obtained. For special cases, their qualitative analysis is given. In the general case, the method of numerical calculation is applied. Curves of motion of the front of destruction are constructed. The importance of the influence of changeability of the module of shift for speed of distribution of front of destruction is revealed.

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Global bifurcation from infinity in nondifferentiable perturbations of half-linear eigenvalue problems for ordinary differential equations of fourth order

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We consider the following nonlinear differential equation

$$\begin{gathered}(p\,(x)\,y'')'' - (q(x)y')' + r(x)y = \lambda\tau(x)\,y\, + \alpha(x)y^{+}(x) + \\ \beta(x)y^{-}(x) + f(x, y, y', y'', y''', \lambda) + g(x, y, y', y'', y''', \lambda),\ x \in (0, l),\end{gathered} \tag{1}$$

subject to boundary conditions

$$y(0) = y'(0) = y(l) = y'(l) = 0,$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, p is a positive twice continuously differentiable function on $[0, l]$, q is a nonnegative continuously differentiable function on $[0, l]$, r, α, β are real-valued continuous functions on $[0, l]$, τ is a positive continuous function on $[0, l]$, is a positive continuous function on $[0, l]$, and $y^{+}(x) = \max\{y(x), 0\}$, $y^{-}(x) = (-\,y(x))^{+}$. The functions f and g are real-valued and continuous on $[0, l] \times \mathbb{R}^5$ and satisfy the following conditions: there exist small positive constant $\varkappa$ and positive constant M such that

$$\left|\tfrac{f(x,y,s,v,w,\lambda)}{y}\right| \leq M,\ x \in [0, l],\ (y, s, v, w) \in \mathbb{R}^4,\ y \neq 0,\ \lambda \in \mathbb{R}; \tag{3}$$

$$g(x, y, s, v, w, \lambda) = o(|y| + |s| + |v| + |w|) \text{ as } |y| + |s| + |v| + |w| \to \infty, \tag{4}$$

uniformly in $(x, \lambda) \in [0, l\,] \times \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$.

Let E be the Banach space $C^3[0, l] \cap (b.c.)$ with the norm $||u||_3 = ||u||_\infty + ||u'||_\infty + ||u''||_\infty + ||u'''||_\infty$, where by $(b.c.)$ we denote the set of functions satisfying the boundary conditions (2), $||u||_\infty = \max\limits_{x\in[0,\,l]} |u(x)|$

It is known [1] that half-linear eigenvalue problem which obtained from (1), (2) by setting $f \equiv 0$ and $g \equiv 0$ has two unbounded sequences of simple half-eigenvalues

$$\lambda_1^+ < \lambda_2^+ < \,\ldots\, < \lambda_k^+ < \,..\text{and } \lambda_1^- < \lambda_2^- < \,\ldots\, < \lambda_k^- < \,\ldots\,.$$

For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ the half-eigenfunction y_k^ν corresponding to the half-eigenvalue λ_k^ν is contained in S_k^ν, where $S_k^\nu \subset E$ is the set of functions that have the nodal properties of eigenfunctions of the linear problem (1), (2) with $f \equiv 0$, $g \equiv 0$, $\alpha \equiv 0$, $\beta \equiv 0$, and their derivatives [2]. Furthermore, aside from solutions on the collection of the half-lines $\{(\lambda_k^\nu, ty_k^\nu) : t > 0\}$ and trivial ones, problem (1), (2) with $f \equiv 0$, $g \equiv 0$, has no other solutions.

Let

$$N_\alpha = \max_{x\in[0,\,l]} |\alpha(x)|,\ N_\beta = \max_{x\in[0,\,l]} |\beta(x)|,\ N_{\alpha,\,\beta,\,M} = N_\alpha + N_\beta + M,$$

and

$$I_k^+ = \left[\lambda_k^+ - \frac{N_{\alpha,\,\beta,\,M}}{\tau_0}, \lambda_k^+ + \frac{N_{\alpha,\,\beta,\,M}}{\tau_0}\right],\ I_k^- = \left[\lambda_k^- - \frac{N_{\alpha,\,\beta,\,M}}{\tau_0}, \lambda_k^- + \frac{N_{\alpha,\,\beta,\,M}}{\tau_0}\right],$$

where $\tau_0 = \min\limits_{x\in[0,\,l]} \tau(x)$.

Problem (1), (2) in the case when g satisfies the condition

$$g(x,y,s,v,w,\lambda) = o(|y|+|s|+|v|+|w|) \text{ as } |y|+|s|+|v|+|w| \to 0,$$

uniformly in $(x,\lambda) \in [0,l\,] \times \Lambda$, for any bounded interval $\Lambda \subset \mathbb{R}$, studied in a recent paper [3], where it is shown that for each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ there is a connected component C_k^ν of the set $\mathcal{D}$ of solutions that contains $I_k^\nu \times \{0\}$, is contained in $(\mathbb{R}\times S_k^\nu) \cup (I_k^\nu \times \{0\})$, and is unbounded in $\mathbb{R}\times E$.

The main results of this note are the following assertions.

Lemma 1. *For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ the set of asymptotic bifurcation points of the problem (1), (2) with respect to the set $\mathbb{R}\times S_k^\nu$ is non-empty. In addition, if (λ,∞) is an asymptotic bifurcation point of problem (1), (2) with respect to $\mathbb{R}\times S_k^\nu$, then $\lambda \in I_k^\nu$.*

Theorem 1. *For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ there exists a connected component D_k^ν of the set $\mathcal{D}$ for which at least one of the followings holds:*

(i) D_k^ν meets $I_{k'}^{\nu'} \times \{\infty\}$ with respect to the set $\mathbb{R}\times S_{k'}^{\nu'}$ for some $(k',\nu') \neq (k,\nu)$;

(ii) D_k^ν meets $\mathbb{R}\times\{0\}$ for some $\lambda \in \mathbb{R}$;

(iii) the projection $P_{R\times\{0\}}(D_k^\nu)$ of the set D_k^ν on $\mathbb{R}\times\{0\}$ is unbounded in $\mathbb{R}\times E$.

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On a mixed problem for heat equation with time deviation in boundary conditions

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In the half-strip $\Pi = \{(x;t): \ 0 < x < 1, \ t > 0\}$ we consider the following mixed problem

$$L\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) u(x,t) = 0, \qquad (x;\, t) \in \Pi, \tag{1}$$

$$u(x,0) = \varphi(x), \quad 0 < x < 1, \tag{2}$$

$$\left. l_j u \right|_{x=0} = 0, \quad t > 0, \quad j = 0,1 \tag{3}$$

$$\left. l_j u \right|_{x=0} = 0, \ 0 < \ t \le \omega, \ j = 2,3 \tag{4}$$

where $Lu = a^2 u_{xx}(x,t) - u_t(x,t)$; $\ l_j u = u(x, t + (1-j)\omega) + \alpha_j u(1-x, \ t + j\omega) \ (j = 0,1)$, $l_j u = a_{j-2} u_x^{(j-2)}(x,t) + b_{j-2} u_x^{(j-2)}(1-x,t) \ (j = 2,3)$, $a, \omega, \ \alpha_j, a_j, b_j \ \ (j = 0,1)$ are real constants , $a \neq 0, \ \omega > 0, \alpha_0 \alpha_1 \neq 0$.

The solution of the problem is the function $u(x,t)$ satisfying the following conditions

1)$u(x,t) \in C^{2,1}(\Pi) \bigcap C(0 < x < 1, \ t \ge 0)$; $\int_0^t u(x,\tau)\, d\tau \in C(0 \le x \le 1, \, t \ge 0)$;

2)$l_j u(x,t) \in C(0 \le x < 1, \ t > 0)$, $j = 0,1$;

3)$l_j u(x,t) \in C(0 \le x < 1, \, 0 < \ t \le \omega)$, $j = 2,3$;

4) $u(x,t)$ satisfies equalities (1)-(4) in the ordinary sense.

The following theorem is proved.

Theorem. *Let* $a_0 b_1 + b_0 a_1 \neq 0$, $\quad \varphi(x) \in C^2[0,1]$ *and* $\left. l_j \varphi(x) \right|_{x=0} = 0 \quad (j = 2,3)$. *Then the problem (1)-(4) has a unique solution represented by the formula:*

$$u(x,t) = \varphi(x) + \frac{1}{\pi i} \int\limits_{\hat{\Gamma}_c} \lambda^{-1} e^{\lambda^2 t} \left[a^2 \int_0^1 G_2(x,\xi,\lambda)\, \varphi''(\xi)\, d\xi - Q(x,\lambda,\varphi(0),\varphi(1)) \right] d\lambda +$$
$$+ \frac{1}{\pi i} \int\limits_{\Gamma_c} \lambda e^{\lambda^2 t} Q(x,\lambda,z_0(\lambda),z_1(\lambda))\, d\lambda,$$

where

$$Q(x,\lambda,\rho,q) = \left(e^{-\frac{\lambda}{a}} - e^{\frac{\lambda}{a}}\right)^{-1} \left[\left(p e^{-\frac{\lambda}{a}} - q\right) e^{\frac{\lambda}{a} x} + \left(q - p e^{\frac{\lambda}{a}}\right) e^{-\frac{\lambda}{a} x} \right],$$

$$z_0(\lambda) = \left(\alpha_1 e^{2\lambda^2 \omega} - \alpha_0\right)^{-1} \alpha_1 e^{\lambda^2 \omega} \left[e^{\lambda^2 \omega} \int_0^\omega e^{-\lambda^2 t} \gamma_0(t)\, dt - \alpha_1 \int_0^\omega e^{-\lambda^2 \omega} \gamma_1(t)\, dt \right],$$

$$z_1(\lambda) = \left(\alpha_1 e^{2\lambda^2 \omega} - \alpha_0\right)^{-1} e^{\lambda^2 \omega} \left[\alpha_1 e^{\lambda^2 \omega} \int_0^\omega e^{-\lambda^2 t} \gamma_1(t)\, dt - \int_0^\omega e^{-\lambda^2 t} \gamma_0(t)\, dt \right],$$

$$\gamma_s(t) = \varphi(s) + \frac{a^2}{\pi i} \int_{\hat{\Gamma}_c} \mu^{-1} e^{\mu^2 t} d\mu \int_0^1 G_1(x, \xi, \mu) \varphi''(\xi) d\xi, \quad (s = 0, 1),$$

$G_1(x,\xi,\mu)$ and $G_2(x,\xi,\lambda)$ are the Green functions of the spectral problems:

$$L\left(\frac{d}{dx}, \mu^2\right) y(x, \mu) = 0, \quad l_j y(x, \mu)|_{x=0} = 0 \quad (j = 2, 3)$$

and

$$L\left(\frac{d}{dx}, \lambda^2\right) z(x, \lambda) = 0, \;\; z(0, \lambda) = z(1, \lambda) = 0,$$

respectively,

$$\Gamma_c = \left\{z: \quad Rez^2 = c, \quad Rez > 0\right\},$$

$$\overset{\wedge}{\Gamma}_c = \left\{z: \quad z = \pm i\sigma, \;\; \sigma \in \left[2c\sqrt{1+\sqrt{2}}, \infty\right]\right\} \bigcup \{z: \quad z = c(1 + i\eta),$$

$$\eta \in \left[-1-\sqrt{2}, \qquad 1+\sqrt{2}\right]\right\},$$

c-is such a positive constant that,

$c > \max\left(c_0, ln\left|\frac{\alpha_0}{\alpha_1}\right|\right)$, $c_0 > |Re\mu_\nu|$, where μ_ν- are the poles of the Green function $G_1(x,\xi,\mu)$, that obviously lie in the finite width strip including the imaginary axis.

Note that the present problem was solved by the combined Laplace transformation method and M.L.Rasulov's residue method [1],[2].

Specific peculiarity of the given statement first of all is the presence of time deviation in boundary conditions that as seems is of mathematical interest and also circles real heat transfer in the rod at where ends temperature is proportionally transferred to the apposite side with some delay $\omega > 0$, and before this moment proportional temperatures and neat-exchange with external medium are maintained.

Secondary, the boundary conditions $l_j u|_{x=0} = 0$ and appropriate requirements 2),3) from the definition of the solution are more natural and expedient since in the usual statement would have been represented in the form:

$$\begin{array}{l} u(0, t + (1-j)\omega) + \alpha_j u(1, t + j\omega) = 0 \quad (t > 0, \;\; j = 0, 1), \\ a_{j-2} u_x^{(j-2)}(0, t) + b_{j-2} u(1, t) = 0 \quad (0 < t \leq \omega, \;\; j = 2, 3) \end{array}$$

and in the definition of the classic solution we would have demand

$u(x,t) \in C(0 \leq x \leq 1, \; t \geq 0) \bigcap C^{1,0}(0 \leq x \leq 1, \; 0 < t < \omega)$,that significantly harrows the class of solvable problems. Because the requirement of continuous adjoining to the boundary of the solution and its derivative to x is a more right adjoining than continuous adjoin to the $x = 0$of the function $l_j u$ linear combinations $u(x, \cdot)$, $u(1-x, \cdot)$ and their derivatives. For example, for

$\varphi(x) = \cos 2\pi x$, $a_0 = -b_0 = a_1 = 1$, $b_1 = 0$, $\alpha_0 \alpha_1 \neq 0$ all the conditions of the above theorem are satisfied, at the same time in particular for $\alpha_0 \neq -e^{4\pi^2 \omega}$, it is easy to see that there is no simply such a solution continuous at the boundary point $(0, \omega)$.

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On properties of weighted modulus continuity in Lebesgue spaces

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In this abstract, we consider a weighted modulus of continuity on Lebesgue spaces. We proved that the modulus of continuity is well-defined in weighted Lebesgue spaces and having all the fine properties of the usual modulus of continuity.

First, we give that the weighted modulus of continuity has properties similar to the usual modulus of continuity defined by

$$\Omega_p\left(f;\delta\right)=\sup_{0<h\leq\delta}\left(\int_0^\infty\left(\frac{|f(x+h)-f(x)|}{\left(1+x^2\right)\left(1+h^2\right)}\right)^p dx\right)^{\frac{1}{p}}$$

We refer the reader to the works [1] and [2].

Let x$\in(0,\,\infty)$ and let $\rho(x)=1+x^2$.

Now we give the main result of this abstract.

Theorem 1. *Let* $1\leq p<\infty$ *and let* $f\in Lp,\rho(0,\infty)$. *Then the following statements hold:*

1. $\sup\limits_{\delta>0}\Omega_p\left(f;\delta\right)\leq\frac{7}{3}\left\|f\right\|_{L_{p,\rho}(0,\infty)}$;

2. $\Omega_p\left(f;0\right)=0$ *and* $\lim\limits_{\delta\to 0}\Omega_p\left(f;\delta\right)=0$;

3. $\Omega_p\left(f;\delta\right)$ *is monotone increasing function.*

4. $\Omega_p(f,\delta)$ *is a continuous function.*

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On embeddings into the total Morrey spaces in the Dunkl setting

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On the $\mathbb{R}^d$ the Dunkl operators $\left\{D_{k,j}\right\}_{j=1}^{d}$ are the differential-difference operators associated with the reflection group $\mathbb{Z}_2^d$ on $\mathbb{R}^d$. We study some embeddings into the total Morrey space (D_k-total Morrey space) $L_{p,\lambda,\mu}(\mu_k)$, $0 \le \lambda, \mu < d + 2\gamma_k$ associated with the Dunkl operator on $\mathbb{R}^d$ introduced by Guliyev in [1] in the Euclidean case, see also [5]. These spaces generalize the Morrey spaces associated with the Dunkl operator on $\mathbb{R}^d$ (D_k-Morrey space) so that $L_{p,\lambda}(\mu_k) \equiv L_{p,\lambda,\lambda}(\mu_k)$ and the modified Morrey spaces associated with the Dunkl operator on $\mathbb{R}^d$ (modified D_k-Morrey space) so that $\widetilde{L}_{p,\lambda}(\mu_k) \equiv L_{p,\lambda,0}(\mu_k)$.

Let $B(x,r) := \{y \in \mathbb{R}^d : |x-y| < r\}$ denote the ball in $\mathbb{R}^d$ that centered in $x \in \mathbb{R}^d$ and having radius $r > 0$, $B_r = B(0,r)$. Then having $\mu_k(B_r) = \int_{B_r} d\mu_k(x) = b_k\, r^{d+2\gamma_k}$. We denote by $L_p(\mu_k) \equiv L_p(\mathbb{R}^d, d\mu_k)$, $1 \le p < \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L_p(\mu_k)} := \Big(\int_{\mathbb{R}^d} |f(x)|^p d\mu_k(x) \Big)^{1/p} < \infty.$$

Definition 1. [2] *Let* $1 \le p < \infty$, $0 \le \lambda \le d+2\gamma_k$ *and* $[t]_1 = \min\{1,t\}$, $t > 0$. *We denote by* $L_{p,\lambda}(\mu_k)$ *Morrey space (*$\equiv$ D_k*-Morrey space), by* $\widetilde{L}_{p,\lambda}(\mu_k)$ *the modified Morrey space (*$\equiv$ *modified* D_k*-Morrey space), associated with the Dunkl operator [2, 3] and by* $L_{p,\lambda,\mu}(\mu_k)$ *Morrey space (*$\equiv$ *total* D_k*-Morrey space) as the set of locally integrable functions* $f(x)$, $x \in \mathbb{R}^d$, *with the finite norms*

$$\|f\|_{L_{p,\lambda}(\mu_k)} := \sup_{x\in\mathbb{R}^d,\, t>0} \left(t^{-\lambda} \int_{B_t} \tau_x |f|^p(y)\, d\mu_k(y) \right)^{1/p},$$
$$\|f\|_{\widetilde{L}_{p,\lambda}(\mu_k)} := \sup_{x\in\mathbb{R}^d,\, t>0} \left([t]_1^{-\lambda} \int_{B_t} \tau_x |f|^p(y)\, d\mu_k(y) \right)^{1/p},$$
$$\|f\|_{L_{p,\lambda,\mu}(\mu_k)} := \sup_{x\in\mathbb{R}^d,\, t>0} \left([t]_1^{-\lambda}\, [1/t]_1^{\mu} \int_{B_t} \tau_x |f|^p(y)\, d\mu_k(y) \right)^{1/p},$$

respectively.

Note that

$$L_{p,0,0}(\mu_k) = \widetilde{L}_{p,0}(\mu_k) = L_{p,0}(\mu_k) = L_p(\mu_k),$$
$$L_{p,\lambda,\lambda}(\mu_k) = L_{p,\lambda}(\mu_k),\ L_{p,\lambda,0}(\mu_k) = \widetilde{L}_{p,\lambda}(\mu_k).$$

Lemma 1. *If* $0 < p < \infty$, $0 \le \lambda \le d + 2\gamma_k$ *and* $0 \le \mu \le d + 2\gamma_k$, *then*

$$L_{p,d+2\gamma_k,\mu}(\mu_k) \subset\!\succ L_\infty(\mathbb{R}^d) \subset\!\succ L_{p,\lambda,|d|}(\mu_k)$$

and

$$\|f\|_{L_{p,\lambda,d+2\gamma_k}(\mu_k)} \le b_k^{1/p} \|f\|_{L_\infty(\mathbb{R}^d)} \le \|f\|_{L_{p,d+2\gamma_k,\mu}(\mu_k)}.$$

Lemma 2. *Let* $1 \le p < \infty$, $0 \le \lambda \le d + 2\gamma_k$ *and* $0 \le \mu \le d + 2\gamma_k$. *Then*

$$L_{p,\lambda,\mu}(\mu_k) = L_{p,\lambda}(\mu_k) \cap L_{p,\mu}(\mu_k)$$

and

$$\|f\|_{L_{p,\lambda,\mu}(\mu_k)} = \max\left\{\|f\|_{L_{p,\lambda}(\mu_k)}, \|f\|_{L_{p,\mu}(\mu_k)}\right\}.$$

On the total D_k-Morrey spaces the following embedding is valid.

Lemma 3. *Let* $0 \le \lambda < d + 2\gamma_k$, $0 \le \mu < d + 2\gamma_k$, $0 \le \alpha < d + 2\gamma_k - \lambda$ *and* $0 \le \beta < d + 2\gamma_k - \lambda$. *Then for* $\frac{d+2\gamma_k-\lambda}{\beta} \le p \le \frac{d+2\gamma_k-\mu}{\beta}$

$$L_{p,\lambda,\mu}(\mu_k) \subset\!\succ L_{1,d+2\gamma_k-\alpha,d+2\gamma_k-\beta}(\mu_k)$$

and for $f \in L_{p,\lambda,\mu}(\mu_k)$ *the following inequality*

$$\|f\|_{L_{1,d+2\gamma_k-\alpha,d+2\gamma_k-\beta}(\mu_k)} \le b_k^{\frac{1}{p'}} \|f\|_{L_{p,\lambda,\mu}(\mu_k)}$$

is valid.

Corollary 1. [3, Lemma 3.4] *Let* $0 \le \lambda < d + 2\gamma_k$ *and* $0 \le \beta < d + 2\gamma_k - \lambda$. *Then for* $p = \frac{d+2\gamma_k-\lambda}{\beta}$

$$L_{p,\lambda}(\mu_k) \subset\!\succ L_{1,d+2\gamma_k-\beta}(\mu_k)$$

and for $f \in L_{p,\lambda}(\mu_k)$ *the following inequality*

$$\|f\|_{L_{1,d+2\gamma_k-\beta}(\mu_k)} \le b_k^{\frac{1}{p'}} \|f\|_{L_{p,\lambda}(\mu_k)}$$

is valid.

Corollary 2. [3, Lemma 3.5] *Let* $0 \le \lambda < d + 2\gamma_k$ *and* $0 \le \beta < d + 2\gamma_k - \lambda$. *Then for* $\frac{d+2\gamma_k-\lambda}{\beta} \le p \le \frac{d+2\gamma_k-\mu}{\beta}$

$$\widetilde{L}_{p,\lambda}(\mu_k) \subset\!\succ \widetilde{L}_{1,d+2\gamma_k-\beta}(\mu_k)$$

and for $f \in \widetilde{L}_{p,\lambda}(\mu_k)$ *the following inequality*

$$\|f\|_{\widetilde{L}_{1,d+2\gamma_k-\beta}(\mu_k)} \le b_k^{\frac{1}{p'}} \|f\|_{L_{p,\lambda}(\mu_k)}$$

is valid.

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On the Dirichlet problem for the non-homogeneous Dirichlet problem for a degenerate fractional order Laplace equation

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In this note, we relate to the weak solvability of the non-homogeneous Dirichlet problem for a degenerate fractional order elliptic equation

$$(A)\begin{cases} (-\Delta)^{\alpha/2}\left(\omega(x)\,(-\Delta)^{\alpha/2}\,u(x)\right)=f \quad \text{on} \quad \Omega\subset\mathbb{R}^n, \\ u\Big|_{\mathbb{R}^n\setminus\Omega}=\phi \end{cases}$$

For that, a list of open problems is declared and sufficient conditions are found in the data of the problem $\Omega,\ \alpha,\ n,$ the weight function $\omega:\ \mathbb{R}^n\to[0,\ \infty)$ and the functions $f,\ \varphi$.

Definition. *Denote the trace space $\mathcal{T}r\left(W^{\alpha}_{p,\ \omega}(\Omega)\right)$ of the functions $W^{\alpha}_{p,\ \omega}(\Omega)$ we denote the class of functions $\varphi:\ \mathbb{R}^n\backslash\Omega\to\mathbb{R}$ such that there exists an extension operator $T:\ \varphi\to\Phi$ from $W^{\alpha}_{p,\ \omega}(\mathbb{R}^n\backslash\Omega)$ to $W^{\alpha}_{p,\ \omega}(\mathbb{R}^n)$ such that $\Phi=\varphi$ on $\mathbb{R}^n\backslash\Omega$.*

Definition By solution of problem (A) we call a function $u\in W^{\alpha}_{2,\omega}(\mathbb{R}^n)$ such that $u-\Phi\in\dot{W}^{\alpha}_{2,\omega}(\Omega)$ for which

$$\iint\limits_{\Omega\times\Omega}\frac{[u(x)-u(y)]\ [\mathrm{g}(x)-\mathrm{g}(y)]}{|x-y|^{n+2\alpha}}\omega(x)\,dxdy=\int_{\Omega}f(x)\Phi(x)\,dx.$$

Theorem 1. *Let $f\in L_2(\Omega)$, $\varphi\in\mathcal{T}r\left(W^{\alpha}_{2,\omega}\ (\Omega)\right)$, $0<\alpha<1,\quad p>1$, $n\geq1$. The weight function $\omega:\ \mathbb{R}^n\to(0,\ \infty)$ satisfies*

$$\omega^{-1}(Q)\leq c\,|Q|^{1-2\alpha/n}$$

for all balls $Q\subset\mathbb{R}^n$. Then for any pare $(f,\ \varphi)$ there exists a unique weak solution of the problem (A).

In the proofs, we have used the weight inequalities of Poincare Sobolev's type [1, 2, 3].

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Numerical solution of the anomalous filtration problem in a nonhomogeneous porous medium

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In this work, the problem of colmatation-suffosion anomalous filtration in a porous medium is posed and numerically solved. The influence of diffusion, colmatation-suffosion effects on the reservoir properties of a porous medium has been established.

Let the study area of the problem consist of $R\{0 \le x \le l\}$. In R, as suspended particles move deeper into the region, they are deposited (colmatation), they are partially separated from the trapped (precipitated) state and further transferred to other pores (suffosion). Taking into account the above, the balance equation can be described as

$$\varepsilon_0 \frac{\partial c}{\partial t} = D \frac{\partial^\beta c}{\partial x^\beta} - \frac{\partial (vc)}{\partial x} + \frac{\partial \varepsilon}{\partial t}, \tag{1}$$

where c is the volumetric concentration of solid particles in the liquid, $\varepsilon_0, \varepsilon$ are initial and current porosities, D is diffusion coefficient, v is filtration velocity, β is the order of the derivative.

Kinetic equation with respect to porosity in the following form [1]

$$\frac{\partial \varepsilon}{\partial t} = \omega_1 \left(\varepsilon_0 - \varepsilon\right) \left|\nabla p\right| - \omega_2 \varepsilon c, \tag{2}$$

where ω_1, ω_2 are coefficients characterizing the intensity of suffosion and clogging of pores, $|\nabla p|$ is pressure gradient modules p.

The filtration velocity component is defined [2]

$$v = -\frac{k(\varepsilon)}{\mu} \frac{\partial^\gamma p}{\partial x^\gamma}, \tag{3}$$

where μ is the viscosity coefficient of the solute, $k(\varepsilon)$ is the permeability coefficient of the porous medium, which, due to colmatation-suffosion effects, γ is the order of the derivative.

The continuity equation is taken in the form

$$\frac{\partial (\rho \varepsilon)}{\partial t} = \frac{\partial \left(\rho v\right)}{\partial x}, \tag{4}$$

where ρ is the density of the liquid.

We use the equations of state for an elastic fluid and an elastic porous medium [2]

$$\rho = \rho_0 \left(1 + \beta_l \left(p - p_0\right)\right), \varepsilon = \varepsilon_0 + \beta_m \left(p - p_0\right) \tag{5}$$

where β_l is the volume compression coefficient of the liquid, β_m is the elasticity coefficient of the medium, ρ_0 is the initial density of the liquid, p_0 is the initial pressure.

From (5) we have

$$\rho\varepsilon = \rho_0\varepsilon_0 + (\rho_0\varepsilon_0\beta_l + \rho_0\beta_m)(p - p_0) + \rho_0\beta_l\beta_m (p - p_0)^2$$

The last term on the right side of this relation can be neglected due to its smallness in comparison with the others. Then it turns out

$$\rho\varepsilon = \rho_0 (\varepsilon_0 + \beta^* (p - p_0)) \tag{6}$$

where $\beta^* = \varepsilon_0\beta_l + \beta_m$ is coefficient of elasticity of the medium.

For the left side of (4), using the generally accepted hypotheses of the elastic filtration regime, we obtain [3]

$$\frac{\partial(\rho\varepsilon)}{\partial t} = \rho_0\beta^*\frac{\partial p}{\partial t}, \tag{7}$$

Using relations (3) and (5) for the simplest case $k(\varepsilon) = k_0\varepsilon \quad (k_0 = const)$ and, taking into account (6), the right side of (4) is transformed as follows

$$\frac{\partial(\rho v)}{\partial x} = -\frac{k_0}{\mu}\rho_0\left((\varepsilon_0 + \beta^* (p - p_0))\frac{\partial^{\gamma+1}p}{\partial x^{\gamma+1}} + \beta^*\frac{\partial p}{\partial x}\frac{\partial^{\gamma}p}{\partial x^{\gamma}}\right). \tag{8}$$

Substituting (7), (8) into (4), we obtain the piezoconductivity equation in the form

$$\frac{\partial p}{\partial t} = \chi\left((\varepsilon_0 + \beta^* (p - p_0))\left(\frac{\partial^{\gamma+1}p}{\partial x^{\gamma+1}}\right) + \beta^*\frac{\partial p}{\partial x}\frac{\partial^{\gamma}p}{\partial x^{\gamma}}\right), \chi = \frac{k_0}{\mu\beta^*}.$$

As in the theory of elastic regime [3] in the last equation, the nonlinear term can be discarded as the second order of smallness, which gives

$$\frac{\partial p}{\partial t} = \chi^*(p)\left(\frac{\partial^{\gamma+1}p}{\partial x^{\gamma+1}}\right), \chi^*(p) = \chi (\varepsilon_0 + \beta^* (p - p_0)). \tag{9}$$

So, we obtain a system of equations for the solute transport, consisting of equations (1), (2), (3) and piezoconductivity equation (9).

System equations (1), (2), (3), (9) is solved on the initial and boundary conditions by the method of finite difference.

Based on the numerical results, suspended particle concentration, medium porosity, filtration velocity and pressure fields were determined. The effects of fractional derivative orders and model parameters on the filtration characteristics of the medium were analyzed.

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Euler type system of equations in variational problems with delayed argument

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We consider a variational problem for a functional depending on one constant delay $h > 0$. Namely, the problems of the following form are studied:

$$J(x(\cdot)) = \int_{t_0}^{t_1} L\left(t,\, x\left(t\right), \dot{x}\left(t\right), x\left(t-h\right), \dot{x}\left(t-h\right)\right) dt \to \min, \tag{1}$$

$$x\left(t\right) = \varphi\left(t\right), \quad t \in \left[t_0 - h,\ t_0\right),\ x\left(t_0\right) = x_0,\ x\left(t_1\right) = x_1 \in R^n\,, \tag{2}$$

where R^n is the n–dimensional Euclidean space, t_0, t_1, x_0 and x_1 are given points, and $t_1 - t_0 > h$; further, the given function $\varphi\left(t\right) : \left[t_0\,,\ t_1\right] \to R^n$ is twice continuous and differentiable. The function, $L\left(t,\, x, \tilde{x}, y, \tilde{y}\right) : \left[t_0,\, t_1\right] \times R^n \times R^n \times R^n \times R^n \to R = \left(-\infty,\, +\infty\right),$ called the integrand, is assumed to be twice continuously differentiable with respect to the set of variables.

The required function $x(\cdot)$ is continuous and the derivatives $\dot{x}(\cdot)$ and $\ddot{x}(\cdot)$ are piecewise smooth. We call such functions $x(\cdot)$ satisfying the boundary condition (2) admissible.

Note that problem (1), (2) was studied in [1-3].

In [1, 3], analogs of the Euler equation were obtained, which is a system of second order neutral type equations with retarded and leading arguments. It is clear that finding a solution to such a system of equations presents difficulties. For this reason, it is relevant to obtain a system of Euler type equations for problem (1), (2), which is a system of neutral type equations with only a retarded argument. This idea, as the main goal, is being implemented in this paper.

Namely, we prove the following statements.

Theorem 1. *Let an admissible function $\bar{x}(\cdot)$ be a solution to problem (1), (2) and be twice continuously differentiable at the points of the set $T \subset \left[t_0,\, t_1\right]$, where $\left[t_0,\, t_1\right] \backslash T$ is a finite set. Then*

(i) *if $t_1 \in \left[t_0,\ \ t_0 + 2h\right]$, then the admissible function $\bar{x}(\cdot)$ taking into account (2) satisfies a system of equations with a retarded argument of the form:*

$$\begin{cases} L_x\left(t\right) - \frac{d}{dt} L_{\dot{x}}\left(t\right) = 0, & t \in \left[t_0,\ \ t_1\right] \cap T, \\ \\ L_y\left(t\right) - \frac{d}{dt} L_{\dot{y}}\left(t\right) = 0, & t \in \left[t_0 + h,\ \ t_1\right] \cap T, \end{cases} \tag{3}$$

where the functions $L_x\left(t\right)$, $L_{\dot{x}}\left(t\right)$, $L_y\left(t\right)$ and $L_{\dot{y}}\left(t\right)$, $t \in \left[t_0,\ \ t_1\right] \cap T$ are calculated along the function $\bar{x}(\cdot)$, taking into account (2);

(ii) *if* $t_1 > t_0 + 2h$, *then the admissible function* $\bar{x}(\cdot)$ *taking into account (2) satisfies the system of equations with a retarded argument of the form*

$$\begin{cases} L_x(t) - \frac{d}{dt} L_{\dot{x}}(t) = 0, & t \in [t_1 - h,\ t_1] \cap T, \\ L_x(t-h) + L_y(t) - \frac{d}{dt}\left[L_{\dot{x}}(t-h) + L_{\dot{y}}(t)\right] = 0, & t \in [t_0 + 2h,\ t_1] \cap T, \\ L_y(t) - \frac{d}{dt} L_{\dot{y}}(t) = 0, & t \in [t_0 + h,\ t_0 + 2h] \cap T, \\ L_x(t) - \frac{d}{dt} L_{\dot{x}}(t) = 0, & t \in [t_0,\ t_0 + h] \cap T. \end{cases} \tag{4}$$

Note that the system of equations of the form (3) and also of the form (4) is a system of equations of the Euler type.

Theorem 2. *Let an admissible function* $\bar{x}(\cdot)$ *be a solution to the system of equations (3) and or (4), and do not satisfy analogs to the of system Euler equations obtained in [3]. Then the admissible function* $\bar{x}(\cdot)$ *cannot be a solution to problem and (1), (2).*

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Fixed-point iteration method for solution first order differential equations with integral boundary conditions

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Problem statement. We consider the system of n-nonlinear coupled differential equations

$$\dot{x}(t) = f(t, x(t)), t \in [0, T] \tag{1}$$

along with boundary conditions

$$Ax(0) + \int_0^T m(t)\, x(t)\, dt = C. \tag{2}$$

Where $A \in R^{n\times n}$ and $m(t) \in R^{n\times n}$ matrices with $\det N \neq 0$, $N = A + B$, $B = \int_0^T m(t)\, dt$; $f : [0, T] \times R^n \to R^n$ is some given continuous function. Let $C([0, T]; R^n)$ be the Banach space of all continuous functions from $[0, T]$ into R^n with the norm $\|x\| = \max\{|x(t)| : t \in [0, T]\}$.

In general the solution of (1)-(2) is characterized by the following:

Green's function. Let us now study the following problem:

$$\dot{x}(t) = y(t),\ t \in [0, T] \tag{3}$$

$$Ax(0) + \int_0^T m(t)\, x(t)\, dt = C. \tag{4}$$

We have that

Lemma 1 [1-2]. For $y \in C([0, T]; R^n)$ the solution of the BVP (3) and (4) is unique and it is given by

$$x(t) = N^{-1}C + \int_0^T G(t, s)\, y(s)\, ds,$$

where

$$G(t, s) = sign(t - s) \begin{cases} N^{-1}\left(A + \int_0^t m(s)\, ds\right),\ 0 \leq s < t, \\ -N^{-1}\int_t^T m(s)\, ds,\ t \leq s \leq T. \end{cases}$$

Proof. For any $t \in [0, T]$ the solution $x = x(\cdot)$ fulfills

$$x(t) = x_0 + \int_0^t y(s)\, ds, \tag{5}$$

where $x_0$is an arbitrary vector. Let us chose x_0 in such a way that $x(t)$ fulfills Equation (4) . There follows

$$(A+B)\,x_0 + \int_0^T m(t)\int_0^t y(s)\,dsdt = C,$$

with implies

$$x_0 = N^{-1}C - N^{-1}\int_0^T\int_t^T m(s)\,dsy(t)\,dt.$$

If we put this value into Equation (5), we get

$$x(t) = N^{-1}C - N^{-1}\int_0^T\int_t^T m(s)\,dsy(t)\,dt + \int_0^t y(s)\,ds. \tag{6}$$

Since the equality

$$E - N^{-1}B = N^{-1}A$$

holds from equation (6) we have

$$x(t) = N^{-1}C + N^{-1}\int_0^t\left(A + \int_0^s m(\tau)\,d\tau\right)y(s)\,ds - N^{-1}\int_t^T\int_s^T m(\tau)\,d\tau y(s)\,ds.$$

Then boundary problem (3)-(4) becomes

$$x(t) = N^{-1}C + \int_0^T G(t,s)\,y(s)\,ds.$$

So that the proof is given.

Iterative methods. The aim of this section is to show that the sequence of functions x_n, which are solutions of

$$\dot{x}_{n+1}(t) = f(t, x_n(t)), \tag{7}$$

subject to the boundary conditions

$$Ax_n(0) + \int_0^T m(t)\,x_n(t)\,dt = C \tag{8}$$

converges to the solution of problem (3)-(4), where f is a nonlinear analytic function and the initial guess function x_0 can be taken as a solution of the initial problem

$$\dot{x}_0(t) = 0, t \in [0,T] \tag{9}$$

$$Ax_0(0) + Bx_0(T) = C. \tag{10}$$

If $C \neq 0$ we have the solution of (1)-(2) in the linear integral form

$$x(t) = N^{-1}C + \int_0^T G(t,s)\,f(s,x(s))\,ds \tag{11}$$

and the sequence of solutions of problem (3)-(4) as

$$x_{n+1}(t) = N^{-1}C + \int_0^T G(t,s)\, f(s, x_n(s))\, ds. \tag{12}$$

Theorem 1. *Let x and x_n, respectively, be the solutions of (1)-(2) and (7)-(8). Assume that f is a nonlinear analytic function. Then, if $MKT < 1$, the sequence of functions x_n converges to the exact solution x in the L_2 norm.*

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On solvability of mixed problem set for a class of equations that change type

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In a rectangular domain of variables t and x by the contour method we study the solvability of a simplest mixed problem for a class of equations with complex-valued coefficients. Note that one of characteristic properties of these equations is the fact that for the equations of corresponding spectral problems, the argument of roots of characteristic polynomial J.Birkhoff are not constant.

In the paper the existence of solition of the following mixed problem is investigated

$$M\left(t,\frac{\partial}{\partial t}\right)U=L\left(x,\frac{\partial}{\partial x}\right)U, 0<t<T,\ \ 0<x<1 \tag{1}$$

$$U(0,x)=\varphi(x) \tag{2}$$

$$U(t,0)=U(t,1)=0 \tag{3}$$

where $M\left(t,\frac{\partial}{\partial t}\right)=\frac{1}{P(t)}\frac{\partial}{\partial t}$,$L\left(x,\frac{\partial}{\partial x}\right)=\frac{1}{(x+b)^2}\cdot\frac{\partial^2}{\partial x^2}$, $b=b_1+ib_2$,$P\left(t\right)=p_1\left(t\right)+ip_2\left(t\right)$, are complex-valued functions, $p_j\left(t\right)\in C[0,\,1]$ $(j=1,\,2)$, $P_1\left(t\right)\neq 0$, $\varphi\left(x\right)-$ is a given function and U (t, x) is a desired function . The final conditions of solvability will be

$$1^0.\int_0^t p_1\left(\tau\right)d\tau\ <0,\ b_1>0\,,\,b_2>0,;$$

$2^0.Re\left(1+b\right)^2+\omega\left(0\right)Im\left(1+b\right)^2\ <\ 0,$ *if* $Im\left[\overline{p}\cdot\int_0^t p\left(\tau\right)d\tau\right]\ \geq\ 0$ *and* $Re\left(1+b\right)^2+\omega\left(T\right)Im\left(1+b\right)^2<0$ *if* $Im\left[\overline{p}\cdot\int_0^t p\left(\tau\right)d\tau\right]<0$ *where* $\omega\left(t\right)=\int_0^t p_2\left(\tau\right)d\tau\cdot\left(\int_0^t p_1\left(\tau\right)d\tau\right)^{-1}$, $\omega\left(0\right)=p_2\left(0\right)\cdot\left(p_1\left(0\right)\right)^{-1}$

$$3^0.\varphi(x)\in C^2[0,\ 1],\varphi(0)=\varphi(1)=0.$$

Note that the condition 1^0 allows one to go beyond the I.G. Petrovsky parabolicity (orwell-posedness) of equation (1) . Obviously, subject to the condition 2^0, the equation (1) isI.G. Petrovsky parabolic to if and only if $Re\left(P\left(t\right)\right)>0$, $0\leq t\leq T$, while under the condition 1^0, Re $P(t)$ can be zero or negative in some part of $(0,\ T]$.The following theorem is valid.

Theorem. *Let conditions*1^0*,* 2^0 *and* 3^0 *be satisfied. Then the problem (1)-(3) has a classical solution* $U\left(t,\,x\right)\in C^{1,2}\left((0;T]\times[0;1]\right)\bigcap C\left([0;T]\times[0;1]\right)$ *representable by the formula (at t > 0)*

$$U\left(t,\,x\right)=\frac{1}{\pi i}\int\lambda e^{\lambda^2\int_0^t P(\tau)\tau}\cdot\left[\int_0^1 G\left(x,\,\xi,\lambda\right)\left(\xi+b\right)^2\varphi\left(\xi\right)d\xi\right]d\lambda$$

where

$$\Gamma = \bigcup_{j=1}^{3} \Gamma_j$$

$$\Gamma_j = \{\lambda : \lambda = r(1+p_j), \, r \geq R\} \ \ (j = 1, 2)$$

$$\Gamma_3 = \{\lambda : \lambda = R(1+i\eta), \, p_1 \leq \eta \leq p_2\} \ ,$$

$$p_j = K_j(t_j) + (-1)^j \delta, \ K_j(t_j) = -\omega(t) + (-1)^j \sqrt{\omega^2(t)+1}, \ (j = 1, 2)$$

$$\omega(t) = \int_0^t p_2(\tau)\, d\tau \cdot \left(\int_0^t p_1(\tau)\, d\tau \right)^{-1}, \, \omega(0) = p_2(0) \cdot (p_1(0))^{-1}$$

$t_1 = T, \ t_2 = 0$ *if* $Im\left[\overline{p} \cdot \int_0^t p(\tau)\, d\tau\right] \geq 0$ *and* $t_1 = 0, \ t_2 = T$ *if* $Im\left[\overline{p} \cdot \int_0^t p(\tau)\, d\tau\right] < 0$, R-*is a rather large,* δ *is sufficiently small positive number.*

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Optimization of the first order partial differential inclusions with the Dirichlet problem

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The article is devoted to the optimization of the gradient differential inclusions (DFIs) on a rectangular area. The discretization method is the main method for solving the proposed boundary value problem. For the transition from discrete to continuous, an especially proven equivalence theorem is provided. To optimize the posed continuous gradient DFIs, a passage to the limit is required in the discrete-approximate problem. Necessary and sufficient conditions of optimality for such problems are derived in the Euler-Lagrange form. The results obtained are based on locally adjoint mappings, being related co-derivative concept.

The argmaximum set is defined as follows: $F_A(u,v_1^*,v_2^*) = \{(v_1,v_2) \in F(u) : \langle v_1,v_1^*\rangle + \langle v_2,v_2^*\rangle. = H_A\left(u,v_1^*,v_2^*\right)\}$, where $F:\mathbb{R}^n \rightrightarrows \mathbb{R}^{2n}$ is a set-valued mapping and H_A is a Hamiltonian function. A set-valued mapping $F^*:\mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ defined by $F^*\left(v_1^*,v_2^*,(\tilde{u},\tilde{v}_1,\tilde{v}_2)\right) = \{u^* : (u^*,-v_1^*,-v_2^*) \in K_F^*\left(\tilde{u},\tilde{v}_1,\tilde{v}_2\right)\}$ we call the locally adjoint mapping (LAM) to convex mapping F at the point $(\tilde{u},\tilde{v}_1,\tilde{v}_2)$, where $K_F^*\left(\tilde{u},\tilde{v}_1,\tilde{v}_2\right)$ is the cone dual to the cone $K_F\left(\tilde{u},\tilde{v}_1,\tilde{v}_2\right)$.
To solve the stated main problem, as mentioned above, we will start with the optimization of the following Dirichlet-type discrete problem:

$$\text{minimize} \sum_{\substack{x_1=1,\dots,T-1\\ x_2=1,\dots,L-1}} g\left(u_{x_1,x_2},x_1,x_2\right),$$

$$(PD) \quad \left(u_{x_1+1,x_2},u_{x_1,x_2+1}\right) \in F\left(u_{x_1,x_2},x_1,x_2\right), H_i = \{0,\dots,T-i\},$$
$$L_i = \{0,\dots,L-i\}, u_{x_1,L} = \alpha_{x_1,}, u_{x_1,0} = \alpha_{x_10}, x_1 \in H_0; u_{0,x_2} = \beta_{0x_2},$$
$$u_{T,x_2} = \beta_{Tx_2}, x_2 \in L_0, \left(\alpha_{00} = \beta_{00}, \alpha_{0L} = \beta_{0L}, \alpha_{TL} = \beta_{TL}, \alpha_{T0} = \beta_{T0}\right); (i=0,1),$$

where $g\left(\cdot,x_1,x_2\right) : \mathbb{R}^1 \to \mathbb{R}^1 \cup \{+\infty\}$ are proper convex functions, $F\left(\cdot,x_1,x_2\right)$ is a set-valued mapping $F(\cdot,x_1,x_2) : \mathbb{R}^1 \rightrightarrows \mathbb{R}^2 (n=1)$, $\alpha_{x_1L},\alpha_{x_10},x_1 \in H_0; \beta_{0x_2},\beta_{Tx_2},x_2 \in L_0$, are fixed numbers, the feasible solution of problem (PD) is denoted by $\{u_{x_1,x_2}\}_{H_0\times L_0} = \{u_{x_1,x_2} : (x_1,x_2) \in H_0 \times L_0\}$.

First, we study a convex Dirichlet-type problem for first-order gradient DFIs of the form

$$\text{minimize } J[u(\cdot,\cdot)] = \iint\limits_D g\left(u\left(x_1,x_2\right),x_1,x_2\right)dx_1dx_2,$$

$$(PC) \quad \nabla u\left(x_1,x_2\right) \in F\left(u\left(x_1,x_2\right),x_1,x_2\right), D = [0,T]\times[0,L],$$

$$u\left(x_1,L\right) = \alpha_L\left(x_1\right), u\left(x_1,0\right) = \alpha_0\left(x_1\right); u\left(0,x_2\right) = \beta_0\left(x_2\right), u\left(T,x_2\right) = \beta_T\left(x_2\right),$$

$$(\alpha_0(0) = \beta_0(0), \alpha_L(0) = \beta_0(L), \alpha_0(T) = \beta_T(0), \alpha_L(T) = \beta_T(L)),$$
$$\nabla u = \text{ gradient } u = (\partial u/\partial x_1, \partial u/\partial x_2).$$

Here $F(\cdot, x_1, x_2) : \mathbb{R}^1 \rightrightarrows \mathbb{R}^2$ is a convex set-valued mapping, $g(\cdot, x_1, x_2)$ is a continuous proper function; and $\alpha_0(x_1), \alpha_L(x_1)$ and $\beta_0(x_2), \beta_T(x_2)$ are absolutely continuous functions, an admissible solution is understood as an absolutely continuous function.

Theorem 1 *Assume that $F(\cdot, x_1, x_2) :: \mathbb{R}^1 \rightrightarrows \mathbb{R}^2$ is a convex set-valued mapping and $g(\cdot, x_1, x_2)$ are convex proper continuous at points of some feasible solution. Then, in order for the solution $\{\tilde{u}_{x_1,x_2}\}_{H_0 \times L_0}$ to be an optimal solution to the problem (PD), it is necessary that there exist a scalar $\alpha \in \{0, 1\}$ and discrete functions $\left\{u^*_{x_1,x_2}\right\}, \left\{\eta^*_{x_1,x_2}\right\}$ not all zeros such that:*

(1) $u^*_{x_1,x_2} + \eta^*_{x_1,x_2} \in F^*\left(u^*_{x_1+1,x_2}, \eta^*_{x_1,x_2+1}; (\tilde{u}_{x_1,x_2}, \tilde{u}_{x_1+1,x_2}, \tilde{u}_{x_1,x_2+1}), x_1, x_2\right)$
$- \alpha\partial g(\tilde{u}_{x_1,x_2}, x_1, x_2)$,

(2) $u^*_{1,x_2} = 0, u^*_{T,x_2} = 0; \eta^*_{x_1,1} = 0, \eta^*_{x_1,L} = 0$

If $\alpha = 1$, then these conditions are also sufficient for the optimality of $\{\tilde{u}_{x_1,x_2}\}_{H_0 \times L_0}$.

Based on Theorem 1 we formulate the sufficient condition of optimality for the problem (PC).

Theorem 2 *For the optimality of the solution $\tilde{u}(x_1, x_2)$ in the convex problem (PC) it is sufficient that there exist a solution $\{u^*(x_1, x_2), \eta^*(x_1, x_2)\}$ of the Euler-Lagrange type adjoint inclusion such that the conditions (i)-(ii) hold:*

(i)$-\frac{\partial u^*(x_1,x_2)}{\partial x_1} - \frac{\partial \eta^*(x_1,x_2)}{\partial x_2} \in F^*(u^*(x_1, x_2), \eta^*(x_1, x_2); (\tilde{u}(x_1, x_2), \nabla\tilde{u}(x_1, x_2)), x_1, x_2)$
$- \partial g(\tilde{u}(x_1, x_2), x_1, x_2)$ a.e $(x_1, x_2) \in D$,
$u^*(T, x_2) = u^*(0, x_2) = \eta^*(x_1, L) = \eta^*(x_1, 0) = 0$,

(ii) $\nabla\tilde{u}(x_1, x_2) \in F_A(\tilde{u}(x_1, x_2), (u^*(x_1, x_2), \eta^*(x_1, x_2)), x_1, x_2)$ a.e $(x_1, x_2) \in D$.

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On the completeness and minimality of double and unitary systems in morrey-type spaces

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In this work, double and unitary systems of functions in Morrey-type spaces $M^{p,\alpha}(-a,a)$ and $M^{p,\alpha}(0,a)$ are considered. A relationship between the completeness and minimality properties of these systems in these spaces is established.

Let us first define the Morrey space $L^{p,\alpha}(a,b)$, $-\infty < a < b < \infty$, $0 \le a \le 1$. It is a Banach space of all measurable functions on(a,b) with the finite norm

$$\|f\|_{L^{p,\alpha}(a,b)} = \sup_{I\subset(a,b)} \left(|I|^{\alpha-1} \int_I |f(t)|^p \, dt \right)^{\frac{1}{p}},$$

where sup in taken over all intervals $I \subset (a,b)$. It is known that $L^{p,\alpha}(a,b)$, $1 \le p < +\infty$, $\alpha \in (0,1)$, is not separable and $C[a,b]$ is not dense in it. Let

$$M^{p,\alpha}(a,b) = \left\{ f \in L^{p,\alpha}(a,b) : \|f(\cdot+\delta) + f(\cdot)\|_{L^{p,\alpha}(a,b)} \to 0, \delta \to 0 \right\}.$$

$M^{p,\alpha}(a,b)$, for $1 \le p < +\infty$,$0 \le \alpha < 1$, is a separable Banach space and $C_0^{\infty}(a,b)$ (the space of all infinitely differentiable functions on (a,b) with compact support) is dense in it. When defining the space $M^{p,\alpha}(a,b)$, the function $f(\cdot)$ is assumed to be extended outside the interval (a,b) by zero.

Consider the following unitary system of functions of the form

$$v_n^{\pm}(t) = a(t)\,\omega_n^{+}(t) \pm b(t)\,\omega_n^{-}(t), t \in [0,a], n \in N, \tag{1}$$

where $a; b; \omega_n^{\pm} : [0,a] \to C$ are some Lebesgue measurable, generally speaking, complex-valued functions on a finite segment $[0,a]$. Let us associate this system with the following double system

$$\{A(t)\,W_n(t)\,; A(-t)\,W_n(-t)\}_{n\in N}, \tag{2}$$

where

$$A(t) = \begin{cases} a(t), t \in [0,a] \\ b(-t), t \in [-a,0) \end{cases}$$

$$W_n(t) = \begin{cases} \omega_n^{+}(t), t \in [0,a] \\ \omega_n^{-}(t), [-a,0) \end{cases}$$

We define the space $(M^{p,\varepsilon}(a,b))'$ associated with $M^{p,\alpha}(a,b)$ and for brevity, denote it by M'. Let S be a unit ball in $M^{p,\alpha}(a,b)$, i.e.

$$S = f \in M^{p,\alpha}(a,b) : \|f\|_{l^{p,\alpha}}(a,b) \le 1.$$

M' is the Banach space of all measurable functions on (a, b) for with the norm

$$\|g\|_M = \sup_{f \in S} \left| \int_a^b fgdt \right| < +\infty$$

The following theorem is true.

Theorem 1. *Let* $\{a\omega_n^+; b\omega_n^- : \forall n\} \subset M^{p,\alpha}(0, a)$, $1 \leq p < +\infty$, $0 < \alpha \leq 1$, *and the double system*

$$V_{n,m} \equiv (A(t) W_n(t); A(-t) W_m(-t)), n, m \in N,$$

is defined by expressions (2). Then this system is complete in $M^{p,\alpha}(-a, a)$ *if and only if the unitary systems* $\{v_n^+\}_{n \in N}$ *and* $\{v_n^-\}_{n \in N}$ *are complete in* $M^{p,\alpha}(0, a)$ *simultaneously.*

Let the system $\{V_{n,n}\}_{n \in N}$ be minimal in $M^{p,\alpha}(-a, a)$ and $\{h_n^+; h_n^-\}_{n \in N} \subset M'(-a, a)$ be a biortogonal system to it. Define

$$\vartheta_k^+(t) \equiv h_k^+(t) + h_k^-(-t), k \in N$$

and

$$\vartheta_k^n(t) \equiv h_k^-(-t) + h_k^-(t), k \in N.$$

The following main theorem is proved.

Theorem 2. *Let* $\{a\omega_n^+; b\omega_n^- : \forall n\} \subset M^{p,\alpha}(0, a)$, $1 \leq p < +\infty$, $0 < \alpha \leq 1$, *and the double system*

$$V_{n,m} \equiv (A(t) W_n(t); A(-t) W_m(-t)), n, m \in N,$$

is defined by expressions (2). Then this system is minimal in $M^{p,\alpha}(-a, a)$ *if and only if the unitary systems* $\{\vartheta_n^+\}_{n \in N}$ *and* $\{\vartheta_n^-\}_{n \in N}$ *are minimal in* $M^{p,\alpha}(0, a)$.

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Uniform convergence of Fourier series expansions in the root functions system of one fourth-order spectral problem

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Consider the following spectral problem

$$y^{(4)}(x) - (q(x)y'(x)) = \lambda y(x),\ x \in (0,1), \tag{1}$$

$$y''(0) = 0, \tag{2}$$

$$Ty(0) - a\lambda y(0) = 0, \tag{3}$$

$$y''(1) - b\lambda y'(1) = 0, \tag{4}$$

$$Ty(1) - c\lambda y(1) = 0, \tag{5}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv y''' - qy'$, $q \in AC([0,1] : (0,+\infty))$, a, b and c are real constants such that $a < 0$, $b < 0$ and $c > 0$.

It follows from [1] that problem (1)-(5) is reduced to a spectral problem for a J-self-adjoint operator in the Pontryagin space $\Pi_3 = L_2(0,1) \oplus \mathbb{C}^3$. Moreover, the system $\{\hat{y}_k\}_{k=1}^{\infty}$, $\hat{y}_k = \{y_k(x),\, m_k,\, n_k, \tau_k\}$, $m_k = ay_k(0)$, $n_k = by'_k(1)$, $\tau_k = cy_k(1)$, of root vectors of the problem (1)-(5) corresponding to the system $\{\lambda_k\}_{k=1}^{\infty}$ of eigenvalues of this problem forms a basis in the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}^3$ with the appropriate inner product.

By $\{\hat{v}_k\}_{k=1}^{\infty}$, $\hat{y}_k = \{y_k(x),\, s_k,\, t_k, h_k\}$, we denote the system in H which adjoint to the system $\{\hat{y}_k\}_{k=1}^{\infty}$.

Let r, l, ξ $(r < l < \xi)$ be arbitrary fixed natural numbers, and let

$$\Delta_{r,\,l,\,\xi} = \begin{vmatrix} s_r & t_r & h_r \\ s_l & t_l & h_l \\ s_\xi & t_\xi & h_\xi \end{vmatrix}.$$

It follows from [2, Theorem 5.3] that if $\Delta_{r,\,l,\,\xi} \neq 0$, then $\{y_k\}_{k=1,\,k\neq r,\,l,\,\xi}^{\infty}$ forms a basis in L_p, $1 < p < \infty$, which is an unconditional basis for $p = 2$. In this case the system $\{u_k\}_{k=1,\,k\neq r,\,l,\,\xi}^{\infty}$ adjoint to the system $\{y_k\}_{k=1,\,k\neq r,\,l,\,\xi}^{\infty}$ is defined by the following formula

$$u_k(x) = \begin{vmatrix} v_k & s_k & t_k & h_k \\ v_r & s_r & t_r & h_r \\ v_l & s_l & t_l & h_l \\ v_\xi & s_\xi & t_\xi & h_\xi \end{vmatrix}.$$

We introduce the notation:

$$\Delta_{f,\,r,\,l,\,\xi} = \begin{vmatrix} (f, y_r) & t_r & h_r \\ (f, y_l) & t_l & h_l \\ (f, y_\xi) & t_\xi & h_\xi \end{vmatrix},$$

where $f \in L_2(0,1)$.

Alongside the boundary-value problem (1)-(5) we will consider the boundary-value problem (1), (2), $Ty(0)=0$, $y''(1)=0$ and $Ty(1)=0$. It follows from [3, Theorems 5.4, 5.5] and [4, Remark 2.1] that the eigenvalues of this problem are positive, simple, except for the first eigenvalue equal to 0 and having geometric multiplicity 2, and form an infinitely increasing sequence $\{\mu_k\}_{k=1}^{\infty}$.

Theorem 1. *Let r, l, ξ $(r < l < \xi)$ be arbitrary fixed natural numbers such that $\Delta_{r,\,l,\,\xi} \neq 0$, and suppose that the Fourier series of a function $f \in C[0,1]$ in the system $\{\phi_k\}_{k=1}^{\infty}$ of eigenfunctions of the problem (1), (2), $Ty(0)=0$, $y''(1)=0$ and $Ty(1)=0$ uniformly converges on $[0,1]$. If $\Delta_{f,\,r,\,l,\,\xi} \neq 0$, then the Fourier series*

$$\sum_{k=1,\,k\neq}^{\infty} (f, u_k) y_k \tag{6}$$

of function $f(x)$ in the system $\{y_k\}_{k=1,\,k\neq r,\,l,\,\xi}^{\infty}$ uniformly converges on $[0,\varrho]$ for each $\varrho \in (0,1)$. If $\Delta_{f,\,r,\,l,\,\xi} \neq 0$, then series (6) uniformly converges on $[0,1]$.

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Adhesion cracks in a composite reinforced with unidirectional fibers under longitudinal shear

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The Ox_1 axis of the Cartesian coordinate system is compatible with the axis of an arbitrary fiber, which has the form of a continuous circular cylinder. Index j denotes objects related to the j fiber, index s denotes binder objects, index t denotes objects associated with the cylindrical region adjacent to the fiber; moreover, the outer and inner radii of the latter are equal to λ_j and λ_j^1, respectively.

Denoting by u_1 and σ_{ij} the coordinates of the displacement vector and the stress tensor, we apply the following reduction [1]:

$$u_1 = u = u(x_2, x_3),\ u_2 = u_3 = 0\ \sigma_{ij} = \sigma_{ij}(x_2, x_3),\ \sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{23} = 0. \tag{1}$$

$$u = \varphi(z) + \overline{\varphi(z)},\ \sigma_{12} - i\sigma_{13} = 2G\varphi'(z). \tag{2}$$

Here $\varphi(z)$ is a piecewise-holomorphic function,

$$\varphi(z) = \begin{cases} \varphi_s(z) \ \text{ if } \ |z_{j,m,n}| > \lambda_j \ \text{ if or all } \ j \ \text{ and } \ m,n = 0 \pm 1, \pm 2 \\ \varphi_{ij}(z_{j,m,n}) + const \ \text{ if } \ h < |z_{j,m,n}| \le \lambda_j, \\ \varphi_{ij}(z_{j,m,n}) + const \ \text{ if } \ h < |z_{j,m,n}| < h, \end{cases} \tag{3}$$

where $z_{j,m,n} = z - e_j - m\omega_1 - n\omega_2$, $z = x_2 + ix_3$.

Using the representations for the quantities $\varphi_s(z)$, $\varphi_{ij}(z_{j,m,n})$, $\varphi_{aj}(z_{j,m,n})$ obtained in [1], from the condition that $\varphi_s(z)$ and $\varphi_{ij}(z_{j,m,n})$ coincide on the common boundary of the domains of definition of these functions and the average condition for stresses, we find a system of algebraic equations:

$$x_{i,1} + e^{i\psi_i} \sum_{j,p\ge 0,q\ge 1} e^{i\psi_j(p+1)} r_{0,p}^{i,j} \left(c_{j,p+1,q} x_{j,q} + d_{j,p+1,q} \overline{x}_{j,q} \right) +$$

$$+ \sum_{j,p\ge 1} \left\{ \xi_{a,j} e^{-i\psi_j} \left[\left(\frac{G_j}{G_s} a_{j,1,p} - c_{j,1,p} \right) x_{j,p} + \left(\frac{G_j}{G_s} b_{j,1,p} - d_{j,1,p} \right) \overline{x}_{j,p} \right] + \right.$$

$$\left. + \Delta e^{i\psi_j} \left(c_{j,1,p} x_{j,p} + d_{j,1,p} \overline{x}_{j,p} \right) \right\} = \frac{<\sigma_{12}> - i <\sigma_{13}>}{2G}$$

$$x_{i,k} + e^{i\psi_i(k+1)} \sum_{j,p\ge 0,q\ge 1} e^{i\psi_j(p+1)} r_{k,p}^{i,j} \left(c_{j,p+1,q} x_{j,q} + d_{j,p+1,q} \overline{x}_{j,q} \right) = 0. \tag{4}$$

where $x_{j,p}$ are free parameters in the representations of work [1] for the values $\varphi_{ij}(z_{j,m,n})$ and $\varphi_{aj}(z_{j,m,n})$, ξ_{aj} is the volumetric content of the j-th fiber, $r^{i,j}_{k,p}$ coefficients at $z^k(p+1)!$.

The remaining quantities are determined from the relations

$$\Delta = \frac{\delta_2\overline{\omega}_1 - \delta_1\overline{\omega}_2}{F}, \quad F = |\omega_1\omega_2| \sin\alpha, \ \delta_k = 2\zeta\left(\frac{\omega_k}{2}\right),$$

$$c_{i,k,p} = g_{1,l}h_i^{k+p} \cdot A_{k,p,l} \quad b_{i,k,p} = g_{2,l}h_i^{p-k} \cdot A_{k,p,l}$$

$$d_{l,k,p} = h_l^{k+p}\left(g_{1,l}B_{k,p,l} - g_{2,l}\delta_k^p\right), \ a_{l,k,p} = g_{2,l}h^{p-k}\left(\delta_k^p + B_{k,p,l}\right),$$

$$g_{1,l} = \frac{1}{1+G_s/G_t}, \ g_{2,l} = \frac{g_{1,l}G_s}{G_t}. \tag{5}$$

$$A_{k,p,l} = \sum_{0\le q\le k} C^{1/2}_{k+p-q}(\cos\theta_l)C_q^{-1/2}(\cos\theta_l),$$

$$B_{k,p,l} = \sum_{\max\{0,k-p+1\}\le q\le k} C^{1/2}_{p-q,q-1}(\cos\theta_l)C_q^{-1/2}(\cos\theta_l)$$

where δ_k^p is the Kronecker symbol, $C_n^{1/2}$ and $C_n^{-1/2}$ are Gegenbauer polynomials.

To find the effective shear moduli G_{12} and G_{13} of the Chentsov coefficients $\mu_{13,12}$ and $\mu_{12,13}$, we use the relation

$$\frac{<\sigma_{12}> - i <\sigma_{13}>}{2G_s} = 2\Delta \sum_{j,p\ge 1} e^{i\psi_j}\left(c_{j,1,p}x_{j,p} + d_{j,1,p}x_{j,p}\right) =$$

$$= \frac{<\sigma_{12}>}{2G_{12}}(1 - i\mu_{13,12}) + \frac{<\sigma_{13}>}{2G_{13}}(\mu_{13,12} - i). \tag{6}$$

Considering, for some $p > 1$, the ordered collections $x_{j,p}$ as elements of a real Banach space (X, v) with the norm $v(x) = \sum\limits_{j,p} |x_{j,p}|\,\rho^p, \ (X = \{x|v(x) < \infty|\})$, we establish that in the absence of touching fibers, system (4) defines a linear injective operator $A : X \to X$; $A = I + U$, where I is the identity operator, U is the limit (in the norm) of a sequence of finite-dimensional operators. Therefore, system (4) has a unique solution in the space X, and to find the latter, we apply the reduction method.

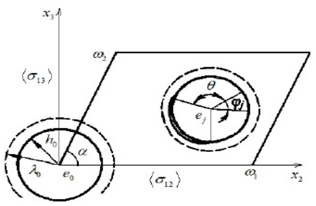

Fig. 1. Geometry of the material and the system of coordinates.

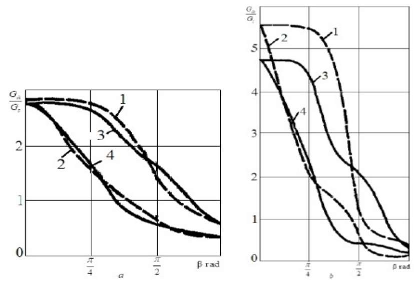

Fig. 2. Dependence of effective shear moduli on half of the crack opening angle β at $\xi = 0,5$ (a) and $\xi = 0,7$ (b): 1, 3) G_{13}/G_s ; 2, 4) G_{12}/G_s (solid lines- hexagonal structure, dotted lines - tetragonal).

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Monitoring of patients after co poisoning

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Along with the diagnosis of poisoning by carbon monoxide poisoning in order to forecast has great importance for the consequences of monitoring. It is to necessary observe during a certain time the health status of persons poisoned. As, carbon monoxide poisoning can cause damage to tissues and organs such as the cardiovascular and respiratory systems, muscles, liver, and kidneys. Major causes of tissue and organ damage are not only hypoxia, but also oxidative stress, formation of oxygen reactive species, neuron necrosis, apoptosis, and abnormal inflammation.

There is limited information regarding cardiac imaging after CO poisoning. One case report involving the use of cardiac magnetic resonance imaging (CMRI) detected late gadolinium enhancement (LGE) during the 4-month follow-up of a patient with severe CO poisoning. Figure 1 shows a representative CMRI image of injury patterns and changes. Another report described a patient with acute subendocardial injury after CO poisoning.

Acute kidney Injury (AKI) is rarely caused by CO poisoning. AKI is a clinical syndrome that decreases renal function, resulting in the retention of nitrogenous waste products and dysregulation of extracellular volume and electrolytes. This case is categorized as stage 3 on the AKI Network.

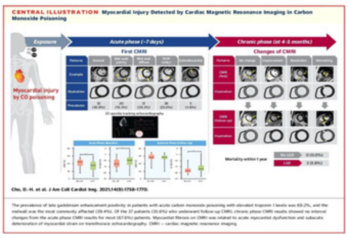

Fig. 1. Representative CMRI image of injury patterns and changes.

In recent years, innovations in the chemical and construction industry for example, streets surrounded by tall and thin skyscrapers on both sides, traffic congestion is reduced with respect to the speed of vehicles, carbon monoxide which is removed from vehicles accumulate in the air near-surface where people breathe in a closed environment and carbon monoxide collected in the atmosphere, in less windy conditions create a dangerous situation for the health of people. All of these lead to chronic intoxication. For these reasons, the following needs to be considered:

1. Differential diagnosis of patients in comatose; 2. Health surveillance of a poisoned person after a certain period of time.

Monitoring needs to be conducted after successful treatment in the hospital. Therefore, starting time of monitoring should coincide with the end of the treatment. Functional parameters and biochemical analysis of carbon monoxide victims need to be examined from time to time during the monitoring (fixed time interval). Particularly, type of poisoning, more affected poisoning of the body, and more nervous and cardiovascular systems, mainly duet of the majority of these indicators are checked by selecting from among the more specific ones. In most cases, the determination of patient treatment verification of indicators reflecting the health of people is selected in the process of stationary treatment. Analysis in the selected interval of time should be checked for prevention and prognosis of consequences after poisoning. 1. it reveals the state of critical or being in the process of change conditions in the status of a patient for whom a plan of future measures will be worked out; 2. it provides data on the previous state giving feedback that will be worked out; relating to earlier successes and failures of a definite policy or programs; 3. it checks on conformity with regulations and contractual obligations;

A need for monitoring a stated problem is determined by a doctor and it depends on the degree of poisoning. Periods may vary in the range of a week, month, quarter, six months, or year. To organize monitoring we shall add a module to the intelligent system for differential diagnosis.

Monitoring will be carried out by the following mathematical methods:

1. Time series or dynamics series is a statistical material on the significance of some parameters (of one parameter in the simplest case) of a process being studied which is collected in different moments of time. Each unit of statistical material is called a measurement or readout. For each readout, the time of measuring or the number of measurements in succession must be given in time series.

2. Time series analysis. Time series analysis presents a body of mathematical-statistical methods of analysis intended for recalling the time series structure or for their prediction.

The time series method is used to observe the trend of change of indicators. The basis of time series analysis is that former happenings have important indications for future happenings. Time series data is a sequence of successive moments of time, which reflects the situation. In contrast to randomly selected analysis, time series analysis is based on observation data of equal time. Time series analysis can be often found in medicine. Time series analysis has two goals: determination of the nature of the queue and prognosis. In both cases, the model must be specified before the turn to the interpretation of the data.

Manna-Whitney's U-criterion, Wilcoxon's T-criterion, Friedman's criterion, and Kraskal-Wallis's H-criterion, which are biostatistical parametric and non-parametric criteria, are used to observe the dynamics of indicators in the time interval. Since the application of the time series method during monitoring reveals whether the change interval of any indicator is within the norm, it is possible to minimize the number of inspections. Studies have shown that periodic analysis reveals the tendency of parameters to change in the process of carbon monoxide poisoning and the most variable in the process of poisoning, are those who are poorly treated. For example, the empirical and calculated regression line (trend) of total protein change over 18 months is given in the graph below (Fig. 2).

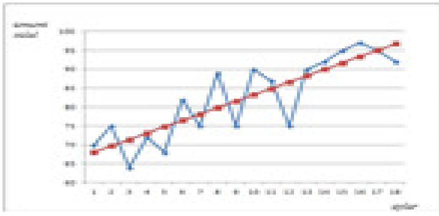

Fig. 2. Dynamics of total protein change

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One analogue of the inversion of the Sobolev embedding theorem

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Let D be the circle $|z| < 1$ in the complex plane. For $0 < p \leqslant \infty$ we denote by $L_p(D)$ the Lebesgue space of complex functions on D with respect to the flat Lebesgue measure with the usual quasi-norm $\|f\|_{L_p(D)}$. Spaces Sobolev $W_p^s(D)$ are well known and quite deeply studied. The following Sobolev embedding theorem is well-known [1]:

$$W_q^1(D) \subset L_p(D),$$

where $2 \leq p < \infty$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{2}$.

It turns out that the following analogue of the inversion of this theorem holds for rational functions of a given degree.

Theorem 1. *Let $p > 2$ and*

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{2}.$$

Then for any rational function r of degree at most n with poles outside the circle D

$$\|r\|_{W_q^1(D)} \leq c\sqrt{n}\|r\|_{L_p(D)}\,,$$

where $c > 0$ and depends only on p.

Note that this relation is exact in the sense of the parameters p and n included in it. Note that the accuracy with respect to the growth of the factor $\sqrt{n}$ is easily confirmed by the example of the functions $r(z) = z^n$, $n = 1, 2, \ldots$. The quasi-norm $\|r\|_{W_q^1}$ cannot be replaced respectively by the quasi-norm $\|r\|_{W_u^1}$ and for no $u > q$. This can be verified by the example of the simplest rational function $r(z) = (z_0 - z)^{-1}$, for $|z_0| > 1$.

It should be noted that various aspects of these relations and their applications were previously studied by the author in [2], [3].

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Free vibrations of inhomogeneous, weakened cylindrical panels in contact with a viscoelastic medium, reinforced by shafts

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The problem was solved based on the application of the Hamilton-Ostrogradsky variation principle. For this, the total energy of the system is written:

$$J = V_p + V_k + V_{1p} + V_{1k} + A_0 \tag{1}$$

Here V_p - is the potential energy of the cylindrical panel, V_k - is the kinetic energy of the cylindrical panel, V_{1p} - potential energy of the shafts, V_{1k} - kinetic energy of the shafts, A_0- is the work done by the forces acting on the orthotropic cylindrical panel by the viscoelastic base in the displacements of the points of the panel. Expressions of energies included in the left side of equation (1) are given in [1]. To take into account the inhomogeneity along the thickness of the cylindrical panel, it was considered that the Young's modulus and the density of the material are a function of the coordinate varying through the thickness [2]. The force q_z acting on the cylindrical cover by the viscoelastic medium is expressed by the deflection $w(x, y, z)$ of the cylindrical cover as follows:

$$q_z = k_\nu w - k_p \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - \int_0^t \Gamma(t-\tau)\, w(\tau)\, d\tau \tag{2}$$

Here k_v Winkler's coefficient, k_p is Pasternak's coefficient, which is found by experiment, t is time, $\Gamma(t-\tau)$ is the viscosity kernel.

1. boundary conditions are also added to the energy expression. In the case of a hinged joint

$$u = v = w = M_x = 0 \;\; x = 0; l \;\; u = v = w = M_x = 0, \;\; \varphi = 0; \varphi_0 \tag{3}$$

It is assumed that the coordinate axes coincide with the main curvature lines of the plate, and the shafts are in rigid contact with the coating along these lines:

$$u_i(x) = u(x, y_i) + h_i \varphi_1(x, y_i), \;\; \vartheta_i(x) = \vartheta(x, y_i) + h_i \varphi_2(x, y_i) \tag{4}$$

$$w_i(x) = w(x, y_i), \;\; \varphi_i(x) = \varphi_1(x, y_i), \;\; \varphi_{kpi}(x) = \varphi_2(x, y_i); \;\; h_i = 0,5h + H_i^1$$

According to the Hamilton-Ostrogradsky variational principle:

$$\delta W = 0$$

Here - $W = \int_{t'}^{t''} Jdt$ is the Hamiltonian effect, and t' - t'' - are given arbitrary moments of time.

$$J = V_k + V_{1k} - V_p - V_{1p} - A_0 \tag{5}$$

Let us find the displacements of the cylindrical panel as follows:

$$\begin{aligned} u &= u_0 \sin\frac{\pi m x}{l} \sin k\frac{\pi\varphi}{\varphi_0} \sin\omega t; \\ \vartheta &= \vartheta_0 \sin\frac{\pi m x}{l} \sin k\frac{\pi\varphi}{\varphi_0} \sin\omega t; \\ w &= w_0 \sin\frac{\pi m x}{l} \sin k\frac{\pi\varphi}{\varphi_0} \sin\omega t. \end{aligned} \tag{6}$$

Here, u_0, v_0, w_0 are the unknown constants, $m,\ k$ are the wave numbers in the length and width directions of the cylindrical panel, respectively.

Using the solutions of (6), we get the following frequency equation as a result:

$$\det\|a_{ij}\| = 0,\ i,\ j = 1,3 \tag{7}$$

Equation (7) is a transcendental equation with respect to its ω frequency. Its roots are calculated by numerical method. Calculations show that the specific oscillation frequencies of the system increase as the number of shafts and the values of the inhomogeneity parameter increase.

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Rayleigh quotient for two-interval periodic Sturm-Liouville problems

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In this at work, we study a spectral problem of new type consisting of a two-interval Sturm-Liouville equation with periodic boundary conditions and two additional interaction conditions specified at the common end of the left and right intervals, the so-called transmission conditions. In particular, we used the well-known Rayleigh quotient to prove an important result related to the eigenvalues.

Main Results. Eigenvalue problems for Sturm-Liouville equations are a class of boundary value problems that are often encountered in solving many problems in mathematical physics using the method of separation of variables, the so-called Fourier's method. There is an extensive literature on the spectral theory of Sturm-Liouville type operators. as well as on the spectral theory of differential operators (see, for example [1], [2], [4] and references cited therein). The central result of Sturm-Liouville theory is that a given function (smooth enough) can be expanded into a series of orthonormal eigenfunctions. This result is used to substantiate the Rayleigh-Ritz minimization principle, which is based on the well-known Rayleigh quotient (see, for example, [1]).

Rayleigh quotient is also used to find eigenvalue estimates, as well as to establish some other spectral results related to Sturm-Liouville problems. In this study, we will use the Rayleigh quotient to study some important properties a new type of Sturm-Liouville problem. Let us consider a new type of periodic Sturm-Liouville problem, consisting of the two-interval differential equation defined two disjoint intervals, given by

$$L(y) := -y''(x) + q(x)y(x) = \lambda y(x),\ x \in [-\pi, 0) \cup (0, \pi] \tag{1}$$

subject to the periodic boundary conditions

$$y(-\pi) = y(\pi), \quad y'(-\pi) = y'(\pi) \tag{2}$$

and two supplementary conditions at the interior singular point $x = 0$, given by

$$y(0+) = y(0^-), \quad y'(0+) = y'(0-) + \alpha y(0-) \tag{3}$$

where $q(x) = \begin{cases} q_1(x) \text{ for } x \in [-\pi, 0) \\ q_2(x) \text{ for } x \in (0, \pi] \end{cases}$, $q_1, q_2(x)$ are real-valued functions, $q_1 \in C[-\pi, 0]$, $q_2 \in C[0, \pi]$, α is a real parameter, λ is a complex spectral parameter. We call this problem the Periodic Sturm-Liouville Boundary Value Transmission Problem (P-SLBVTP, for short).

Theorem 1. *All eigenvalues of the P-SLBVTP (1)-(3) are real.*

Proof. Let $(\lambda, y(x))$ be any eigenpair of the P-SLBVTP (1)-(3), where $\lambda \in \mathbb{C}$. Since the potential $q(x)$ is a real-valued function and α is a real number, we get upon taking the complex conjugate of the equalities (1)-(3) that $(\overline{\lambda}, \overline{y(x)})$ is also an eigenpair of the same problem (1)-(3). By Green's second identity, we have

$$(\overline{\lambda}-\lambda)(\int_{-\pi}^{0-}|y(x)|^2dx+\int_{0+}^{\pi}|y(x)|^2dx)=W(y,\overline{y};\pi)-W(y,\overline{y};-\pi)$$

$$-W(y,\overline{y};0+)-W(y,\overline{y};0-) \tag{4}$$

where $W(f,g;x)=f(x)g'(x)-f'(x)g(x)$. By the boundary-transmission conditions (2)-(3) the right hand side of (4) is zero. Since the eigenfunction $y(x)$ is not identically zero this means $\lambda=\overline{\lambda}$, i.e. the eigenvalue λ is real.

Corollary 2. *Let $(\lambda, y(x))$ be any eigenpair. Then by the previous theorem $(\overline{\lambda}, \overline{y(x)})$ is also an eigenpair. Hence $z=y(x)+\overline{y(x)}$ is also a real-valued eigenfunction corresponding to the same eigenvalue λ. Therefore, without loss of generality, we can assume that all eigenfunctions of the P-SLBVTP (1)-(3) are real-valued.*

Theorem 3. *If $(\lambda_1, y_1(x))$ and $(\lambda_2, y_2(x))$ are two eigenpairs and $\lambda_1 \neq \lambda_2$, then the eigenfunctions $y_1(x)$ and $y_2(x)$ are orthogonel in the Hilbert space $L_2[-\pi,0]\oplus L_2[0,\pi]$.*

Proof By using Green's second identity and the boundary-transmission conditions (2)-(3) as in the proof of the previous Theorem 1 we can show that

$$(\lambda_1-\lambda_2)\langle y_1,y_2\rangle_{L_2[-\pi,0]\oplus L_2[0,\pi]}=0$$

That is $y_1(x)$ and $y_2(x)$ are orthogonel.

Theorem 4. *If $\alpha>0$ and $q(x)\geq 0$ then all eigenvalues of the P-SLBVTP (1)-(3) are nonnegative.*

Proof Let $(\lambda_0, \varphi_0(x))$ be any eigenpair of the P-SLBVTP (1)-(3). Then by using the Rayleigh quotient (see, [1]) we can prove that

$$\lambda_0=\frac{1}{\parallel\varphi_0\parallel^2_{L_2[-\pi,0]\oplus L_2[0,\pi]}}(\int_{-\pi}^{0^-}[(\frac{d\varphi_0(x)}{dx})^2+q(x)\varphi_0^2(x)]dx$$

$$+\int_{0^+}^{\pi}[(\frac{d\varphi_0(x)}{dx})^2+q(x)\varphi_0^2(x)]dx)+\alpha y^2(0+)\geq 0$$

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On solvability of third-order operator-differential equation with discontinuous coefficient and operator in the boundary condition

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Let H be a separable Hilbert space with the scalar product (x, y), $x, y \in H$.
Consider, in the space H, the boundary value problem

$$u'''(t) + \rho(t)A^3u(t) = f(t), \quad t \in \mathbb{R}_+, \tag{1}$$

$$u'(0) = Su(0), \tag{2}$$

where A is a self-adjoint positive-defined operator ($A = A^* > cE$, $c > 0$, E is the identity operator), $S \in L\left(H_{5/2}, H_{3/2}\right)$, $f(t) \in L_2(\mathbb{R}_+; H)$, $u(t) \in W_2^3(\mathbb{R}_+; H)$, $\rho(t) = \alpha$, if $0 \le t \le 1$, $\rho(t) = \beta$, if $t > 1$, and α, β, are positive numbers, in general not equal to each other. Here H_γ is the scale of Hilbert spaces generated by the operator A ($H_\gamma = D(A^\gamma)$, $\gamma > 0$, $(x, y)_\gamma = (A^\gamma x, A^\gamma y)\,,\ x, y \in D(A^\gamma)$), and $L(X, Y)$ is the set of linear bounded operators acting from a Hilbert space X into another Hilbert space Y,

$$L_2(\mathbb{R}_+; H) = \left\{ f(t) : \|f\|_{L_2(\mathbb{R}_+;H)} = \left(\int\limits_0^{+\infty} \|f(t)\|_H^2\, dt \right)^{1/2} < +\infty \right\},$$

$$W_2^3(\mathbb{R}_+; H) = \Big\{ u(t) :\ u'''(t) \in L_2(\mathbb{R}_+; H)\,,\ A^3u(t) \in L_2(\mathbb{R}_+; H)\,,$$

$$\|u\|_{W_2^3(\mathbb{R}_+;H)} = \left(\|u'''\|^2_{L_2(\mathbb{R}_+;H)} + \left\|A^3u\right\|^2_{L_2(\mathbb{R}_+;H)} \right)^{1/2} \Big\}.$$

Note that derivatives are understood in the sense of the theory of distributions in a Hilbert space [1].

Assuming that

$$W_{2,S}^3(\mathbb{R}_+; H) = \left\{ u(t) :\ u(t) \in W_2^3(\mathbb{R}_+; H)\,, u'(0) = Su(0) \right\},$$

denote by P_0 the operator acting from the space $W_{2,S}^3(\mathbb{R}_+; H)$ to the space $L_2(\mathbb{R}_+; H)$ by the rule

$$P_0u(t) = u'''(t) + \rho(t)A^3u(t),\ u(t) \in W_{2,S}^3(\mathbb{R}_+; H)\,.$$

Theorem. *Let $A = A^* \ge cE$, $c > 0$, $S \in L\left(H_{5/2}, H_{3/2}\right)$, and the operator*

$$S_{\alpha,\beta} = E + \frac{1}{\sqrt[3]{\alpha}}A^{-1}S + \frac{\sqrt[3]{\alpha} - \sqrt[3]{\beta}}{\sqrt[3]{\beta} + \sqrt[3]{\alpha}\omega_1}\omega_2 \left(E - \frac{\omega_2}{\sqrt[3]{\alpha}}A^{-1}S \right) e^{-\sqrt[3]{\alpha}(\omega_1+1)A} +$$

$$+\frac{\sqrt[3]{\alpha}-\sqrt[3]{\beta}}{\sqrt[3]{\beta}+\sqrt[3]{\alpha}\omega_2}\omega_1\left(E-\frac{\omega_1}{\sqrt[3]{\alpha}}A^{-1}S\right)e^{-\sqrt[3]{\alpha}(\omega_2+1)A}$$

has a bounded inverse in the space $H_{3/2}$. Then the operator P_0 isomorphically maps the space $L_2(\mathbb{R}_+;H)$ onto the space $W^3_{2,S}(\mathbb{R}_+;H)$.

Note that the boundary value problem (1), (2) for $S=0$ was investigated in [2].

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Investigation of parabolic fractional integral operators in parabolic local generalized Morrey spaces

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Let P be a real $n\times n$ matrix, whose all eigenvalues have a positive real part, $A_t = t^P$, $t > 0$, $\gamma = trP$ is the homogeneous dimension on R^n and Ω is an A_t- homogeneous of degree zero function, integrable to a power $s > 1$ on the unit sphere generated by corresponding parabolic metric.

Definition 1. *Let $\varphi(x,r)$ be a positive measurable function on $R^n\times(0,\infty)$ and $1 \le p < \infty$. For any fixed $x_0 \in R^n$ we denote by $LM^{\{x_0\}}_{p,\varphi,P} \equiv LM^{\{x_0\}}_{p,\varphi,P}(R^n)$ the parabolic generalized local Morrey space, the space of all functions $f \in L^{loc}_p(R^n)$ with finite quasinorm*

$$\|f\|_{LM^{\{x_0\}}_{p,\varphi,P}} = \|f(x_0+\cdot)\|_{LM_{p,\varphi,P}}.$$

Also by $WLM^{\{x_0\}}_{p,\varphi,P} \equiv WLM^{\{x_0\}}_{p,\varphi,P}(R^n)$ we denote the weak parabolic generalized local Morrey space of all functions $f \in WL^{loc}_p(R^n)$ for which

$$\|f\|_{WLM^{\{x_0\}}_{p,\varphi,P}} = \|f(x_0+\cdot)\|_{WLM_{p,\varphi,P}} < \infty.$$

Definition 2. *Let $S_\rho = \{w \in R^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in R^n $(n \ge 2)$ equipped with the normalized Lebesgue surface measure $d\sigma$ and Ω be A_t-homogeneous of degree zero, i.e. $\Omega(A_t x) \equiv \Omega(x)$, $x \in R^n$, $t > 0$. The parabolic fractional integral operator $I^P_{\Omega,\alpha}f$ with rough kernels, $0 < \alpha < \gamma$, of a function $f \in L^{loc}_1(R^n)$ is defined by*

$$I^P_{\Omega,\alpha}f = \int_{R^n} \frac{\Omega(x-y)\,f(y)}{\rho(x-y)^{\gamma-\alpha}}dy.$$

We prove the boundedness of the parabolic integral operator $I^P_{\Omega,\alpha}$ with rough kernel from one parabolic local generalized Morrey space $LM^{\{x_0\}}_{p,\varphi_1,P}(R^n)$ to another one $LM^{\{x_0\}}_{q,\varphi_2,P}(R^n)$, $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{\gamma}$, and from the space $LM^{\{x_0\}}_{1,\varphi_1,P}(R^n)$ to the weak space $WLM^{\{x_0\}}_{q,\varphi_2,P}(R^n)$, $1 \le q < \infty$, $1 - \frac{1}{q} = \frac{\alpha}{\gamma}$.

Theorem. *Suppose that $x_0 \in R^n$, $0 < \alpha < \gamma$ and the function $\Omega \in L_{\frac{\gamma}{\gamma-\alpha}}(S_\rho)$ is A_t-homogeneous of degree zero. Let $1 \le p < \frac{\gamma}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\gamma}$, and the pair (φ_1,φ_2) satisfy the condition*

$$\int_r^\infty \frac{\underset{t<\tau<\infty}{ess\sup}\varphi_1(x_0,\tau)\,\tau^{\frac{n}{p}}}{t^{\frac{\gamma}{q}+1}}dt \le C\varphi_2(x_0,r),$$

where C does not depend on x_0 and r. Then the operator $I^{P}_{\Omega,\alpha}$ is bounded from $LM^{\{x_0\}}_{p,\varphi_1,P}$ to $LM^{\{x_0\}}_{q,\varphi_2,P}$ for $p > 1$ and from $LM^{\{x_0\}}_{1,\varphi_1,P}$ to $WLM^{\{x_0\}}_{q,\varphi_2,P}$ for $p = 1$.

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On basicity of perturbed exponential system in Morrey-type spaces

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An exponential system with a piecewise linear phase depending on some parameters is considered in this work. The basicity of this system is studied in a subspace of Morrey space where continuous functions are dense.

Let us first define the Morrey space on the unit circle $\gamma = \{z \in C : |z| = 1\}$ on the complex plane C; $\omega = int\gamma$ will denote the unit ball in C.

$L^{p,\alpha}(\gamma)$, $1 \le p < +\infty$, $0 \le \alpha \le 1$, will denote the normed space of all measurable functions $f(\cdot)$ on γ with the finite norm

$$\|f\|_{L^{p,\alpha}(\gamma)} = \sup_B \left(|B \cap \gamma|_\gamma^{\alpha-1} \int_{B\cap\gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty,$$

($|B \cap \gamma|_\gamma$ is the linear measure of intersection $B \cap \gamma$), where sup is taken over all balls centered at γ with an arbitrary positive radius. $L^{p,\alpha}(\gamma)$ is a Banach space with respect to this norm. We also define the space $L^{p,\alpha}(-\pi,\pi)$, $1 \le p < +\infty$, $0 \le \alpha \le 1$, which consists of measurable functions $f(\cdot)$ on $(-\pi,\pi)$ with the finite norm

$$\|f\|_{L^{p,\alpha}(-\pi,\pi)} = \sup_{I\subset[-\pi,\pi]} \left(|I|^{\alpha-1} \int_I |f(t)|^p \, |dt| \right)^{1/p} < +\infty,$$

where sup is taken over all intervals $I \subset [-\pi,\pi]$. It is not difficult to see that the correspondence $f(t) =: F(e^{it})$, $t \in (-\pi,\pi)$, $F(\cdot) \in L^{p,\alpha}(\gamma)$ establishes an isometric isomorphism between the spaces $L^{p,\alpha}(\gamma)$ and $L^{p,\alpha}(-\pi,\pi)$. Therefore, in what follows we will equate these spaces and denote $L^{p,\alpha}$ with the norm $\|\cdot\|_{p,\alpha}$.

We will consider the subspace $M^{p,\alpha}$ of functions $f(\cdot)$ the shifts of which are continuous in $L^{p,\alpha}$, i.e.

$$\|f(\cdot+\delta) - f(\cdot)\|_{p,\alpha} \to 0, \quad \delta \to 0.$$

Consider the following exponential system

$$E_\beta = \left\{ e^{i(nt - \beta|t| \, sign\, n)} \right\}_{n\in Z}.$$

Using the method of boundary value problems we established the basicity of this system in Morrey spaces $M^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha < 1$. We followed the techniques used in [1-3] and proved the following main theorem:

Theorem 1. *The system E_β forms a basis for the space $M^{p,\alpha}, 1 < p < +\propto$, for $\forall\beta \in C$.*

Attention should be paid to the fact that the corresponding result in case of the system

$$e_\beta \equiv \left\{e^{i(n+\beta sign\, n)t}\right\}_{n\in Z},$$

is quite different. Namely, the basicity of e_β requires inequality type restriction on β.

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The effect of electrokinetic properties on fluid flow in porous media

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In most hydrocarbon fields, the fluid that is injected for displacement flows non-stationary. This is partly determined by the start-up time of wells, the timing of fluid intervals across the cross section in responding wells and other factors. A more rigorous and selective system of intervention is required to improve the sweep efficiency of the producing reservoir in heterogeneous conditions. Oil flow rates, fluid characteristics, geology, reservoir and water permeability systems and optimum reservoir energy production all influence the amount of oil delivered to producing wells - all have a more significant impact in such fields. One of the key challenges in oil production is managing the physicochemical properties of the water used in the creation of oil fields. An evaluation of the effects of electrokinetic factors on various interphase processes in fluid flow in reservoir systems and their function in the above-mentioned flow situations can be recognized as relevant to the investigation of the listed cases.

The author has evaluated approaches to controlling electrokinetic processes in viscous-granular fluids [1]. Waters with these electrokinetic characteristics make a contribution to the filtration process in several ways. In reservoirs with poor permeability, the electrokinetic process imposes an additional resistance to fluid flow, which is characterized by the effect of electro-viscosity. If the electrokinetic effect is neglected, the fluid filtration rate cannot be accurately measured. The influence of the electrokinetic process on the liquid intensifies and the electro viscosity increases as the CEL thickness approaches the radius of the pore channel.

In a reservoir with low permeability, the flow rate can decrease by up to 30% when the water salinity is about 10.5 mol/l. The greatest effect of the electrokinetic phenomenon, or increase in electro viscosity, is felt at higher water salinities for highly permeable media. These explanations allow the possibility of modifying the electrokinetic characteristics of the fluid flow process in porous media by mixing electrolytes with water. At the same time, depending on the choice of electrolyte, electro-viscosity or slip effects in the flow can be achieved. It is well known that in electro-viscosity the electro-potential parameter is the important one. Here, was explained why viscosity changes based on the electro-potential value and provided the formula for the corresponding evaluation. This expression is written as following:

$$\frac{\mu}{\mu_e} = \left(1 - \frac{\beta_* \varphi^2}{2\pi^2 r^2 m}\right)^{-1}$$

Fig. 1 and 2 illustrate the evaluations that were made based on potential and capillary radius changes. Non-linear changes of $\frac{\mu}{\mu_e}$ ratio depending on these parameters is observed in the figures.

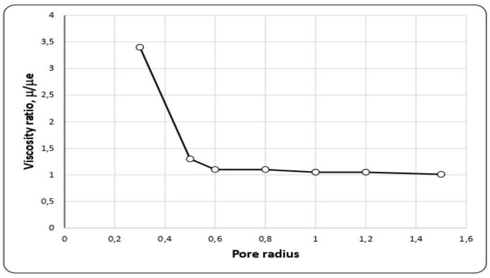

Fig 1. Changing the $\frac{\mu}{\mu_e}$ ratio depending on the pore radius

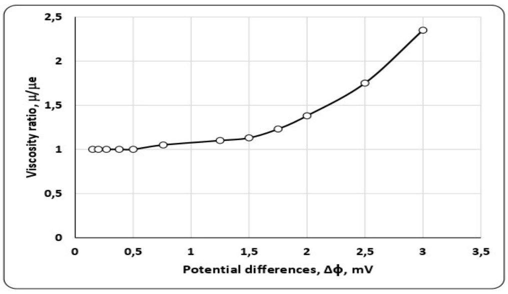

Fig 2. Changing the viscosity $\frac{\mu}{\mu_e}$ratio depending on the potential differences

In particular, an increase in the concentration of dissolved components in the surface layer leads to an increase in viscosity [2, 3]. This can be one of the determining prerequisites for the hydrocarbon displacement process. According to the research carried out, it is possible to control the fluid filtration process in porous media by adding an electrolyte to the water. The intake characteristics of water injection wells can be altered by changing the electrophysical and electrokinetic properties of the fluids injected into the reservoir.

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Description of dissipative extensions of one operator with potential in the form of a Dirac function

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Denote by $D(A_0) = \left\{\psi \in L_2 \bigcap C| - \Delta\psi \in L_2(E_3), \psi(\xi_k) = 0, k = \overline{1,n}\right\}$, the domain of the operator A_0, generated by the expression $-\Delta$.

The paper describes all dissipative extensions of the minimal operator generated by the differential expression

$$-\Delta + \sum_{s=1}^{n} \delta(x - \xi_s),$$

where $\delta(x) -$ the Dirac function, $\xi_s \in E_3$, $\xi_s \neq \xi_{s'}$, when $s \neq s'$.

Denote by J_λ the class of all linear nonexpanding operators V from $n_\lambda = L_2 \oplus [(A_0 - \lambda I) D(A_0)]$ to $n_{\overline{\lambda}}$, not necessarily defined on the whole n_λ. The following theorem is proved

Theorem 1. *For any linear nonexpanding operator V from J_λ all dissipative extensions of the operator A_0 are given by the formulas*

$$\psi = \psi_0 + \sum_{s=1}^{n} \beta_s \varphi_s - \sum_{s=1}^{n} \alpha_s f_s,$$

$$A_V\psi = -\Delta\psi + \sum_{s=1}^{n} (\alpha_s - \beta_s) \delta(x - \xi_s) + \left(\lambda - \overline{\lambda}\right) \sum_{s=1}^{n} (\beta_s \varphi_s + \alpha_s f_s),$$

$$\psi_0 \in D(A_0), \alpha_s \in R, s = \overline{1,n}, \beta_s = \sum_{k=1}^{n} \alpha_k V_{ks},$$

$V_{ks}-$ coefficients of the expansion of V_{ks} according to the system $\{\varphi_k\}_{k=1}^{n}$, $f_k = \frac{e^{i\sqrt{\lambda}|x-\xi_k|}}{4\pi|x-\xi_k|}$, $\varphi_k = \frac{e^{i\sqrt{\overline{\lambda}}|x-\xi_k|}}{4\pi|x-\xi_k|}$, $k = \overline{1,n}$, $\lambda-$any non-real number from the lower half-plane.

Next, for any dissipative extension, we write out the explicit form of the resolvent and prove the following theorem.

Theorem 2. *The resolvent R_λ^V of any dissipative extension of the operator A_0 is an integral operator with a Carleman-type kernel and the kernel $G_V(x, y, \lambda)$ has the form*

$$G_V(x, y, \lambda) = \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} + \sum_{j=1}^{n} \sum_{m=1}^{n} D_m^{(j)}(\lambda) \frac{e^{i\sqrt{\lambda}|y-\xi_m|}}{4\pi|y-\xi_m|} \cdot \frac{e^{i\sqrt{\lambda}|x-\xi_j|}}{4\pi|x-\xi_j|},$$

where $D_m^{(j)}(\lambda) -$ is a meromorphic function of, which has a special form.

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Regularization of necessary conditions of a nonlocal boundary value problem for a three-dimensional mixed-composite equation by new method

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The present work is devoted to the study of a boundary value problem for a three-dimensional mixed-composite type equation with nonlocal boundary conditions. Necessary conditions are derived both in the composite and in the hyperbolic cases and then are regularized by an original method.

A mixed-composite type equation is considered in a bounded three-dimensional domain $D = \{x = (x_1, x_2, x_3) \in R^3\}$convex along Ox_3 axis with Lyapunov boundary Γ:

$$\frac{\partial}{\partial x_3}\left(sign\, x_3 \frac{\partial^2 u(x)}{\partial x_3^2} + \left(\frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} \right)\right) = 0,\ x_3 \neq 0, \tag{1}$$

with non-local boundary conditions:

$$\sum_{m=0}^{2}\sum_{n=1}^{2}\sum_{k=0}^{2}\sum_{1\le p_1\le p_2\le\ldots p_k\le 3} \alpha_{i,p_1p_2\ldots p_k}^{(m,n)}(x')\frac{\partial^k u_n(x)}{\partial x_{p_1}\partial x_{p_2}\ldots\partial x_{p_k}}\Big|_{x_3=\gamma_m(x')} = f_i(x'), \tag{2}$$

$$x' \in S,\ i = \overline{1,6},$$

$$u_1(x',0) = u_2(x',0) = f_0(x'),\ x' \in L = \bar{\Gamma}_1 \bigcap \bar{\Gamma}_2, \tag{3}$$

where $S = proj_{Ox_1x_2} D$, Γ_1 and Γ_2 are upper and lower half-surfaces of Γ: $\Gamma_k = \{\xi = (\xi_1, \xi_2, \xi_3) :\ \xi_3 = \gamma_k(\xi'),\ \xi' = (\xi_1, \xi_2) \in S\}$; functions $\gamma_k(\xi')\,, k = 1, 2,$ are twice-differentiable in S, $\gamma_1(\xi') > 0, \gamma_2(\xi') < 0,\ \xi' \in S$.

The fundamental solution for equation (1) in case $x_3 > 0$ has the form [1]:

$$U_1(x-\xi) = -\frac{1}{4\pi}\ln|x_3 - \xi_3 + |x-\xi||\ . \tag{4}$$

and in case $x_3 < 0$

$$U_2(x-\xi) = \frac{\theta(x_3 - \xi_3 - |x'-\xi'|)}{2\pi}\ln\left|(x_3-\xi_3) + \sqrt{(x_3-\xi_3)^2 - |x'-\xi'|^2}\right|\ . \tag{5}$$

By means of multiplying (1) by the corresponding fundamental solution ((4) or (5)) and its partial derivatives and integrating by parts there were found **regular first and second necessary conditions** for equation (1) both for the composite and hyperbolic case.

The third necessary conditions are singular ($1 \le i \le j \le 3$; $1 \le m \le p \le 3$; $k = 0,\, 2$):

$$\frac{\partial^2 u_1(\xi)}{\partial \xi_i \partial \xi_j}\left|_{\xi_3=\gamma_k(\xi')}\right. =$$

$$= \int_S \frac{1}{|x'-\xi'|^2} \sum_{\substack{p,q=1\\ p\le q}}^{3} (-1)^{k+1} K^{(k)}_{ij,pq}(x',\xi') \frac{\partial^2 u_1(x)}{\partial x_p \partial x_q}\left|_{x_3=\gamma_k(x')}\right. \frac{dx'}{\cos(\nu, x_3)} + ..., \qquad (6)$$

$$\frac{\partial^2 u(\xi)}{\partial \xi_i \partial \xi_j}\left|_{\xi_3\in\Gamma_k}\right. = \int_S \frac{\theta((\gamma_k(x')-\gamma_k(\xi'))-|x'-\xi'|)}{|x'-\xi'|^2}\times$$

$$\times \sum_{\substack{m,p=1\\ m\le p}}^{3} M^{(k)}_{ij,mp}(x',\xi') \frac{\partial^2 u_2(x)}{\partial x_m \partial x_p}\left|_{x_3=\gamma_k(x')}\right. \frac{dx'}{\cos(\nu_x,\, x_3)} + \dots \,. \qquad (7)$$

Building linear combinations of singular third necessary conditions (6) and (7), transforming and grouping the terms in the integrands so that to substitute boundary conditions (2) the third necessary conditions (6) and (7) are **regularized**. These regularized relationships together with boundary conditions (2) let to prove Fredholm property of the above-stated problem (1)-(3).

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On scaling the representativeness of sample statistics in solving a stochastic control problem

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A stochastic problem with conflicting criteria of product quality and feasibility of a production task [1] is formulated by us as:

$$u^m(t_i) = arg \max_{u \in (u_{min}, u_{max})} \cdot$$

$$\cdot \left[F[X_i, u(t_i)] - \lambda(t_i) \cdot \varepsilon \cdot \beta \left(u - \frac{G - \sum_{i=1}^{i-1} u^m(t_i)(t_j - t_{j-1})}{T - t_{i-1}} \right)^2 \right], \tag{1}$$

where the function $\lambda(t_i)$ $i = \overline{1, N}$ from the point of view of the cumulative solution should turn out to be approximations of the main problem.

Here ε is the coefficient of criteria normalization; β weight coefficients.

An approximation of the feedback depth control function based on a simulation model can be expressed as:

$$\lambda(t) = \frac{\delta'}{r} \left(1 - \frac{t}{T} \right); \;\; \delta' = 2 \left| \frac{u^m_{\lambda=0} - \frac{G}{T}}{u^m_{\lambda=0} + \frac{G}{T}} \right|; \quad r \in [r_{min}, r_{max}], \tag{2}$$

where $u^m_{\lambda=0}$ is the value that delivers the maximum to expression (1) in the absence of feedback (ie, with $\lambda(t_i) \equiv 0$), r is some indicator of sample representativeness.

Based on general grounds, the dependence of estimates of statistical parameters on the amount of data is of order $\frac{1}{\sqrt{n}}$.

However, the error of approximation of distribution functions can be significant, especially when it comes to samples of small sizes, so the accuracy of estimating the representativeness of observations up to a certain constant factor in this case is of fundamental necessity.

Thus, the study of the accuracy of solutions to formulations (1) is akin to the general problem of estimating the representativeness of samples for a given volume n.

The question of the measure of discrepancy between the sample distribution function $\Phi_N(x)$ and the distribution function corresponding to the general population $\Phi_G(x)$ was covered by A.N. Kolmogorov in the form of minimizing the modulus of the difference of these functions [2]. In such a measurement, which is usually referred to as nonparametric analysis, it turns out that the process of approximating the sample function to its general analogue obeys a certain distribution $K(t)$, in which the quantity $\frac{1}{\sqrt{n}}$ determines the order of convergence:

$$\forall\, t > 0 : \lim_{t\to\infty} P\left(D_n \le \frac{1}{\sqrt{n}}t\right) = K(t);$$

$$D_n = \sup_{x\in R} \left|\Phi_n(x) - \Phi_n(G)\right|, \tag{3}$$

where $\Phi_n(x)$, $\Phi_G(x)$ are the functions given above, respectively, for the sample size n and for the general population, D_n the maximum deviation of the compared functions, which falls at some point $x \in R^1$, $K(t)$ is of the A.N. Kolmogorov function:

$$K(t) = \sum_{j=-\infty}^{+\infty} (-1)^j e^{-2j^2t^2} \tag{4}$$

Using the main provision of the named theorem on the relationship between the quantities $\frac{1}{\sqrt{n}}$, D_n it is possible to scale the representativeness of the sample data. Based on the differential function $K'(t)$ we find that the maximum probability is reached at the point $t = 0.74$.

Based on this maximum probability, we arrive at the result:

$$r(n) = \frac{1}{0,74}\sqrt{\frac{n}{N}}$$

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On determining the number of loading cycles to failure of car axle

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It is known that structural details subject to repeated variable loading are destroyed at stresses significantly less than the ultimate strength of the material under static loading [1]. The number of loading cycles to failure of details of modern high-speed machines is measured in several million.

We consider the cyclic failure of the rotating axles of a railway car. When a railway car moves its axle rotates while being under the action of a load. We neglect some changes in the axle load and consider it constant. Under the action of this load, there arise variable stresses in the circular cross-section of the rotating axle of the railway car. Satisfaction of the equilibrium equations and corresponding boundary conditions allows determining the stress at an arbitrary point of the cross-section of the axle by the formula:

$$\sigma_{bend} = \frac{M_{bend}}{S} r \sin\theta. \tag{1}$$

Here σ_{bend} is bending stress; M_{bend} is a bending moment; S is a moment of inertia of the axle cross-section; (r, θ) are polar coordinates.

Let the axle under consideration rotate with constant angular velocity ω. This time $\theta = \omega \cdot t$, where t is time, formula (1) goes to the formula:

$$\sigma_{bend} = \frac{M_{bend}}{S} r \sin\omega t. \tag{2}$$

The problem is in finding the numbers of loading cycles to failure of the considered axle of a railway car subjected to the cyclic action of stress (2).

To this end, we determine the difference of maximum and minimum stresses. In the case of the considered case, the marked difference is independent of the number of loading cycles:

$$\sigma_* = \sigma_{\max} - \sigma_{\min} = \frac{2M_{bend}}{S} r. \tag{3}$$

Let N_1 be the number of cycles to the first failure of the axle of the railway car, N_0 be the number of cycles to failure (discontinuity) of the marked axle. This time the values of N_1 and N_0 are determined by the following formulas [2]:

$$N_i = N_{is} \exp\left[\alpha_i \left(1 - \frac{\sigma_*}{\sigma_s}\right)\right] \quad (i = 0, 1) \tag{4}$$

Here N_{0s} and N_{1s} are experimentally determined number of cycles to failure and to first axle damage for $\sigma_* = \sigma_s$; α_0, α_1 are constants and are also experimentally determined according to the technique represented in [2].

The value of σ_s is selected from the change range of σ_*.

Allowing for (3), formulas (4) take the form:

$$N_0 = N_{0s} \exp\left[\alpha_0 \left(1 - \frac{2M_{bend}}{S\sigma_s} r\right)\right] \tag{5}$$

$$N_1 = N_{1s} \exp\left[\alpha_1 \left(1 - \frac{2M_{bend}}{S\sigma_s} r\right)\right] \tag{6}$$

Formulas (5) and (6) determine the number of cycles to failure and to first damages of the railway to failure and to first damages of the railway car axle at the point r. As can be seen from these formulas, the first damages and failures of the axle appear on the axle surface, i.e. for $r = R$, where R is the radius of the axle under consideration.

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On embedding theorems in generalized grand weighted Sobolev spaces

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This abstract is devoted to investigations of grand weighted Sobolev spaces

$$\bigcap_{i=0}^{n} L_{p)}^{\langle l^i \rangle}(G, b_i),$$

where $G \subset R^n$ is the bounded domain, $1 < p < \infty$, $b_i = b_i(x)$ are weighted functions in G, $l^i = \left(l_1^i, l_2^i, \ldots, l_n^i\right)$, $l_j^0 \geq 0$, $l_j^i \geq 0$, $l_i^i > 0$ $(i \neq j,\ j = 1, 2, \ldots, n; \quad i = 1, 2, \ldots, n)$ are whole numbers, which are an indicator of the differential properties of this spaces, all do not simultaneously lie on the $n-$ dimensional plane. First, we introduce a generalized grand weighted Sobolev space and by using integral representation for functions defined on dimensional domains satisfying the $\psi-$horn condition, embedding theorems for the latter spaces are proved.

Definition. *A generalized grand weighted Sobolev spaces* $\bigcap_{i=0}^{n} L_{p)}^{\langle l^i \rangle}(G, b_i)$, *is the closure of the set sufficiently smooth finite functions* $f = f(x)$ *in* R^n *with respect to the*

$$\sum_{i=0}^{n} \left\| b_i(\cdot)\, D^{l^i} f(\cdot) \right\|_{L_{p)}(G)} < \infty,$$

where

$$\|b_i(\cdot)\, f(\cdot)\|_{L_{p)}(G)} = \sup_{0 \leq \varepsilon < p-1} \left(\frac{\varepsilon}{|G|} \int_G |b(x)\, f(x)|^{p-\varepsilon} dx \right)^{\frac{1}{p-\varepsilon}}.$$

Note that in case $l^0 = (0, 0, \ldots, 0)$, $l^i = (0, \ldots, 0,\ l_i, 0, \ldots, 0)$, $b_i(x) = b(x)$ $(i = 0, 1, \ldots, n)$ the space $\bigcap_{i=0}^{n} L_{p)}^{\langle l^i \rangle}(G, b_i)$ coincides with the grand weighted Sobolev space $W_{p)}^{l_1, l_2, \ldots, l_n}(G, b)$ and more over $b_i(x) = 1 (i = 0, 1, \ldots, n)$, then of the space $\bigcap_{i=0}^{n} L_{p)}^{\langle l^i \rangle}(G, b_i)$ coincides with the grand Sobolev spaces $W_{p)}^{l_1, l_2, \ldots, l_n}(G)$, which was studied in the work [1,2].

The following theorems on the properties of functions from the constructed spaces are proved:

1. $D^v : \bigcap_{i=0}^{n} L_{p)}^{\langle l^i \rangle}(G, b_i) \to L_{q-\varepsilon}(G, b)$
2. $D^v : \bigcap_{i=0}^{n} L_{p)}^{\langle l^i \rangle}(G, b_i) \to \bigcap_{i=0}^{n} L_{q-\varepsilon}^{\langle m^i \rangle}(G, g)$.

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The structure of bifurcation components of the solution set of some nonlinear eigenvalue problems

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We consider the following nonlinear eigenvalue problem

$$y^{(4)} - (qy')' = \lambda y + f(x, y, y', y'', y''', \lambda),\ 0 < x < l, \tag{1}$$

$$y''(0) = 0,\ Ty(0) - b\lambda y(0) = 0,\ \ y''(1) = 0,\ Ty(1) - d\lambda y(1) = 0, \tag{2}$$

where $\lambda \in \mathbb{R}$ is an eigenvalue parameter, $Ty \equiv y''' - qy'$, $q(x)$ is positive and absolutely continuous function on $[0, 1]$, b, d are real constants such that $b > 0$, $d < 0$, the function $f \in C([0, 1] \times \mathbb{R}^5)$ is real-valued and satisfies the following conditions:

$$\liminf_{|y|\to 0} \frac{f(x, y, s, v, w, \lambda)}{y} = \underline{f}_0 \in \mathbb{R},\ \ \limsup_{|y|\to 0} \frac{f(x, y, s, v, w, \lambda)}{y} = \overline{f}_0 \in \mathbb{R}, \tag{3}$$

$$\liminf_{|s|\to +\infty} \frac{f(x, y, s, v, w, \lambda)}{y} = \underline{f}_\infty \in \mathbb{R},\ \limsup_{|y|\to +\infty} \frac{f(x, y, s, v, w, \lambda)}{y} = \overline{f}_\infty \in \mathbb{R}, \tag{4}$$

uniformly in $x \in [0, 1]$ and $(s, v, w, \lambda) \in \mathbb{R}^4$.

The eigenvalues of the linear problem (1), (2) with $f \equiv 0$ are nonnegative and simple, and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ (see [1]).

Let E be a Banach space $C^3[0, l] \cap \{y''(0) = y''(1) = 0\}$ with the norm $||y||_3 = \sum_{i=0}^{3} ||\, y^{(i)}||_\infty$, where $||y||_\infty = \max_{x\in[\,0,1]} |y\,(x)|$. For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ let $S_k^\nu \subset E$ be the set defined in [2] and possessing the oscillatory properties of the eigenfunctions (and their derivatives) of the linear problem (1), (2) with $f \equiv 0$.

Theorem 1. *For each $k \in \mathbb{N}$ and each $\nu \in \{+\,,\,-\}$ there exists a union D_k^ν of components of the set of nontrivial solutions of problem (1), (2) which meets $[\lambda_k - \overline{f}_0, \lambda_k - \underline{f}_0] \times \{0\}$ and $[\lambda_k - \overline{f}_\infty, \lambda_k - \underline{f}_\infty] \times \{\infty\}$, and lies in $\mathbb{R} \times S_k^\nu$.*

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On the weak solvability of a nonlocal boundary value problem for the Laplace equation in an unbounded domain

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A nonlocal problem for the Laplace equation in an unbounded domain is considered. A weak solution of this problem is determined in weighted Sobolev spaces generated by a weighted mixed norm. The correct solvability of this problem is proved by the Fourier method. In the classical formulation, this problem was previously considered by E.I. Moiseev [1]. We also note that the same type of problem was considered in the work of M.E. Lerner and O.A. Repin [2]. In a weak formulation, this problem is apparently considered for the first time.

Let $\alpha = (\alpha_1; \alpha_2) \in Z_+^2$ be a multi-index and $|\alpha| = \alpha_1 + \alpha_2$. Accept $\partial^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$. Let $\nu : J \to \overline{R}_+ = [0, +\infty]$ be the weight function, i.e. $\left|\nu^{-1}\{0; +\infty\}\right| = 0$, where $|M|$ is the Lebesgue measure of the set $M \subset R$, R is the real axis, $J = (0, 2\pi)$. $\overline{M}$ is a closure of the set M. $u/_M$ is the trace (if any) of u to M. Put

$$\Pi = \left\{(x; y) \in R^2 : x \in J; y \in R_+\right\}; \; J_0 = \{(0; y) : y \in R_+\};$$
$$J_{2\pi} = \{(2\pi; y) : y \in R_+\}.$$

Denote by $C_0^\infty(\Pi)$ the following space of test functions

$$C_0^\infty(\Pi) = \left\{\varphi \in C^\infty\left(\overline{\Pi}\right) : \varphi/_{\partial\Pi} = 0 \,\&\, \exists C_\varphi > 0 : \varphi(x; y) \equiv 0, \, \forall y \geq C_\varphi\right\}.$$

We also set

$$C_{J_0}^\infty\left(\overline{\Pi}\right) = \left\{\varphi \in C^\infty\left(\overline{\Pi}\right) : \varphi/_{J \bigcup J_{2\pi}} = 0 \& \exists y_\varphi > 0 : \varphi(x; y) \equiv 0, \forall y \geq y_\varphi\right\},$$
$$C_0^\infty[0, +\infty) = \left\{\psi \in C^\infty[0, +\infty) : \psi(0) = 0 \,\&\, \exists C_\psi > 0 : \psi(y) = 0, \, \forall y \geq C_\psi\right\}.$$

Denote by $C_0^\infty(0, +\infty)$ the space of infinitely differentiable and finite functions in $(0, +\infty)$.

The weighted Lebesgue space $L_{p,\nu}(\Pi)$ is defined by the following mixed norm

$$\|f\|_{L_{p,\nu}(\Pi)} = \int_0^{+\infty} \left(\int_0^{2\pi} |f(x; y)|^p \nu(x)\, dx\right)^{\frac{1}{p}} dy, \; 1 \leq p < +\infty.$$

The corresponding Sobolev space is $W_{p,\nu}^m(\Pi)$ with norm

$$\|f\|_{W_{p,\nu}^m(\Pi)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L_{p,\nu}(\Pi)}.$$

We also denote by $L_{p,\nu}(J)$ and $W_{p,\nu}^{m}(J)$ the Lebesgue and Sobolev spaces with norms

$$\|f\|_{L_{p,\nu}(J)} = \left(\int_J |f|^p \nu dx\right)^{\frac{1}{p}},$$

and

$$\|f\|_{W_{p,\nu}^{m}(J)} = \sum_{k=0}^{m} \left\|f^{(k)}\right\|_{L_{p,\nu}(J)},$$

respectively.

We need a class of Mackenhoupt weights $A_p(J)$, that are periodically extended to the entire axis with period 2π and satisfy the condition

$$\sup_{I\subset R}\left(\frac{1}{|I|}\int_I \nu(t)\,dt\right)\left(\frac{1}{|I|}\int_I |\nu(t)|^{-\frac{1}{p-1}}\,dt\right)^{p-1} < +\infty,\ 1<p<+\infty,$$

where sup is taken over all intervals $I \subset R$.

Consider the (so far formal) following nonlocal problem

$$\Delta u(x;y) = 0,\ (x;y)\in \Pi, \tag{1}$$

$$u/_{J_0} = u/_{J_{2\pi}};\ u'_x/_{J_0} = g;\ u/_J = f. \tag{2}$$

Let $\varphi \in C_{J_0}^{\infty}\left(\overline{\Pi}\right)$ be an arbitrary function. Assume $\Pi_\varphi = J\times(0, y_\varphi)$. Multiplying the equation by φ and integrating over Π, we have

$$0 = \int_\Pi \Delta u\varphi dxdy = \int_{\Pi_\varphi} \Delta u\varphi dxdy = /\text{ Gauss-Ostrogradsky formula } / =$$

$$= -\int_{\Pi_\varphi} \nabla u\nabla\varphi dxdy + \int_{\partial\Pi_\varphi} \varphi\frac{\partial u}{\partial n}dl = -\int_\Pi \nabla u\nabla\varphi dxdy + \int_0^{+\infty} \varphi(0;y)\,g(y)\,dy \Rightarrow$$

$$\int_\Pi \nabla u\nabla\varphi dxdy = \int_0^{+\infty} \varphi(0;y)\,g(y)\,dy,\ \forall\varphi\in C_{J_0}^{\infty}\left(\overline{\Pi}\right), \tag{3}$$

where $\frac{\partial u}{\partial n}$ is the derivative with respect to the outward normal to $\partial\Pi$.

Thus, we accept the following

Definition 1. *We will say that the function $u \in W_{p,\nu}^{1}(\Pi)$ is a solution to the problem (1), (2) if it's the restriction to $\partial\Pi$ satisfies the relations*

$$u/_{J_0} = u/_{J_{2\pi}};\ u/_J = f,$$

and the relation (3) holds.

In this case, the function u will be called a weak solution to the problem (1), (2).

We prove the following theorem on the uniqueness of the weak solution of the problem (1), (2).

Theorem 1. *Let $\nu \in A_p(J)$, $1 < p < +\infty$, and $f \in W_{p,\nu}^{1}(J) : f(0) = f(2\pi) = 0$. Then, if the problem (1), (2) has a weak solution for $g \in L_1(J_0)$, then it is unique.*

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On commutator of Hausdorff operators in weighted Lebesgue spaces

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In this abstract, we establish two-weight norm inequalities for the commutator of the one-dimensional Hausdorff operator in the scale of weighted Lebesgue spaces. We also study a similar problem for the dual operator of the commutator of the one-dimensional Hausdorff operator.

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Nodal solutions of some fourth order nonlinear boundary value problems

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We consider the following nonlinear boundary value problem

$$(p(x)y'')'' - (q(x)y')' = \varrho\,\tau(x)g(y(x)),\ x \in (0,l), \tag{1}$$

$$\begin{array}{l} y'(0)\cos\alpha - (py'')(0)\sin\alpha = 0, \\ y(0)\cos\beta + Ty(0)\sin\beta = 0, \\ y'(l)\cos\gamma + (py'')(l)\sin\gamma = 0, \\ y(l)\cos\delta - Ty(l)\sin\delta = 0, \end{array} \tag{2}$$

$Ty \equiv (py'')' - qy'$, $p \in C^2([0,l];(0,+\infty))$, $q \in C^1([0,l];[0,+\infty))$, $\tau \in C([0,l];(0,+\infty))$, ϱ is a positive parameter, and $\alpha,\ \beta, \gamma,\ \delta$ are real constants such that $0 \le \alpha,\ \beta, \gamma,\ \delta \le \frac{\pi}{2}$ (the cases $\alpha = \gamma = 0$, $\beta = \delta = \pi/2$ and $\alpha = \beta = \gamma = \delta = \pi/2$ are excluded). The nonlinear term $g \in C^0\left([0,l] \times \mathbb{R}^5;\mathbb{R}\right)$ and satisfies the following conditions:

(B1) $sg(s) > 0$ for any $s \in \mathbb{R}$, $s \neq 0$;

(B2) there exist positive constants g_0 and g_∞ such that

$$g_0 = \lim_{|s|\to 0} \frac{g(s)}{s},\ g_\infty = \lim_{|s|\to+\infty} \frac{g(s)}{s}.$$

Note that similar nonlinear Sturm-Liouville boundary value problems were studied in [1, 2] (see also references therein).

Along with the boundary value problem (1), (2), consider the following linear eigenvalue problem for the equation

$$(p(x)y'')'' - (q(x)y')' = \lambda\,\tau(x)y(x),\ x \in (0,l), \tag{3}$$

under boundary conditions (2), where $\lambda \in \mathbb{C}$ is a spectral parameter. By Theorems 5.4 and 5.5 of [3] the eigenvalues of the problem (3), (2) are positive, simple and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$. Moreover, for each $k \in \mathbb{N}$ the eigenfunction y_k corresponding to the eigenvalue λ_k has exactly $k-1$ simple zeros in $(0,l)$.

The following theorem is the main result of this note.

Theorem 1. *Let for some $k \in \mathbb{N}$ either the condition*

$$\frac{\lambda_k}{g_0} < \varrho < \frac{\lambda_k}{g_\infty}\ \ or\ \ \frac{\lambda_k}{g_\infty} < \varrho < \frac{\lambda_k}{g_0},$$

holds. Then there are two solutions y_k^+ and y_k^- of the problem (1), (2) such that y_k^+ has exactly $k-1$ simple zeros in $(0,l)$ and is positive near $x = 0$, and y_k^- has exactly $k-1$ simple zeros in $(0,l)$ and is negative near $x = 0$.

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Global regularity in Orlicz-Morrey spaces of solutions to parabolic equations with VMO coefficients

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Introduction. Motivated by the extension of parabolic Calderón-Zygmund theory to the Orlicz context, we study the boundedness of parabolic singular and nonsingular integral operators, and the associated commutators with VMO functions, on generalized Orlicz-Morrey spaces. We also show some applications for strong solutions to nondivergence parabolic equations with VMO coefficients.

We consider the space of functions with bounded mean oscillations (BMO class), the space of functions from BMO with vanishing mean oscillation (VMO class), the generalized Orlicz-Morrey spaces, and the relations between them. Some important applications of the VMO class to the theory of partial differential equations (parabolic equations) with discontinuous coefficients, in nondivergence and divergence form, are then presented. By applying some estimates for parabolic singular integral operators and commutators, we obtain regularity results for these equations.

We show continuity in generalized Orlicz-Morrey spaces $M^{\Phi,\varphi}$ of sublinear integral operators generated by parabolic Calderón-Zygmund operator and their commutators with BMO functions. As a consequence, we obtain a global $M^{\Phi,\varphi}$-regularity result for the linear uniformly parabolic equations with vanishing mean oscillation (VMO) coefficients (see [1, 3]).

Main Results. In the following, besides the standard parabolic metric $\varrho(x) = \max(|x'|, |t|^{1/2})$ we use the equivalent one $\rho(x) = \left(\frac{|x'|^2+\sqrt{|x'|^4+4t^2}}{2}\right)^{1/2}$ introduced by Fabes and Riviére in [2]. The induced by its topology consists of ellipsoids

$$\mathcal{E}_r(x) = \left\{ y \in \mathbb{R}^{n+1} : \ \frac{|x'-y'|^2}{r^2} + \frac{|t-\tau|^2}{r^4} < 1 \right\}, \ |\mathcal{E}_r| = Cr^{n+2}.$$

A function $\Phi : [0,\infty) \to [0,\infty]$ is called a Young function if Φ is convex, left-continuous, $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ and $\lim\limits_{r\to\infty} \Phi(r) = \infty$. For a Young function Φ, the set

$$L^{\Phi}(\mathbb{R}^{n+1}) = \left\{ f \in L^{1,\text{loc}}(\mathbb{R}^{n+1}) : \int_{\mathbb{R}^{n+1}} \Phi(k|f(x)|)dx < \infty \text{ for some } k > 0 \right\}$$

is called the Orlicz space.

Throughout this paper, the following notations will be used:
$x = (x',t), y = (y',\tau) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+$, $\mathcal{E}_r(x) = \{y \in \mathbb{R}^{n+1} : \frac{(x_1-y_1)^2}{r^2} + \ldots + \frac{(x_n-y_n)^2}{r^2} + \frac{(t-\tau)^2}{r^4} < 1\}$.

We now define generalized Orlicz-Morrey spaces of the third kind, see, for example, [3]. The *generalized parabolic Orlicz-Morrey space* $M^{\Phi,\varphi}(\mathbb{R}^{n+1})$ *of the third kind* is defined as the set of all measurable functions f for which the norm

$$\|f\|_{M^{\Phi,\varphi}(\mathbb{R}^{n+1})} \equiv \sup_{x\in\mathbb{R}^{n+1},\, r>0} \varphi(x,r)^{-1}\, \Phi^{-1}\Big(|\mathcal{E}_r(x)|)^{-1}\Big)\, \|f\|_{L^{\Phi}(\mathcal{E}_r(x))}$$

is finite.

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$-domain and $Q = \Omega \times (0,T)$, $T > 0$ be a cylinder in $\mathbb{R}^{n+1}_+$.

The generalized Sobolev-Orlicz-Morrey space $W^{2,1}M^{\Phi,\varphi}(Q)$ consists of all Sobolev functions $u \in W^{2,1}L^{\Phi}(Q)$ with distributional derivatives $D_t^l D_x^s u \in M^{\Phi,\varphi}(Q)$, endowed with the norm

$$\|u\|_{W^{2,1}M^{\Phi,\varphi}(Q)} = \|D_t u\|_{M^{\Phi,\varphi}(Q)} + \sum_{0\le|s|\le 2} \|D_x^s u\|_{M^{\Phi,\varphi}(Q)}.$$

$$\overset{\circ}{W}{}^{2,1}M^{\Phi,\varphi}(Q) = \big\{u \in W^{2,1}M^{\Phi,\varphi}(Q):\ u(x) = 0, x \in \partial Q\big\}, \|u\|_{\overset{\circ}{W}{}^{2,1}M^{\Phi,\varphi}(Q)} = \|u\|_{W^{2,1}M^{\Phi,\varphi}(Q)},$$

where ∂Q means the parabolic boundary $\Omega \cup (\partial\Omega \times (0,T))$.

We consider the Cauchy-Dirichlet problem for the linear parabolic equation

$$u_t - a^{ij}(x) D_{ij} u(x) = f(x) \ \text{ a.a. } x \in Q, \quad u \in \overset{\circ}{W}{}^{2,1}M^{\Phi,\varphi}(Q)\,. \tag{1}$$

where the coefficient matrix $\mathbf{a}(x) = \{a^{ij}(x)\}_{i,j=1}^n$ satisfies

$$\begin{cases} \exists\ \Lambda > 0:\ \Lambda^{-1}|\xi|^2 \le a^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \ \text{ for a.a. } x \in Q,\ \forall \xi \in \mathbb{R}^n \\ a^{ij}(x) = a^{ji}(x) \text{ that implies } a^{ij} \in L_\infty(Q). \end{cases} \tag{2}$$

Theorem 1. [Main result] *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $\mathbf{a} \in VMO(Q)$ satisfy () and $u \in \overset{\circ}{W}{}^{2,1}L^{\Phi}(Q)$ be a strong solution of (). If $f \in M^{\Phi,\varphi}(Q)$ with $\varphi(x,r)$ being measurable positive function satisfying*

$$\int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)\Big(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi(x,s)}{\Phi^{-1}\big(s^{-n-2}\big)}\Big)\Phi^{-1}\big(t^{-n-2}\big)\, \frac{dt}{t} \le C\,\varphi(x,r), \quad (x,r) \in Q \times \mathbb{R}_+\,, \tag{3}$$

then $u \in \overset{\circ}{W}{}^{2,1}M^{\Phi,\varphi}(Q)$ and

$$\|u\|_{\overset{\circ}{W}{}^{2,1}M^{\Phi,\varphi}(Q)} \le C\|f\|_{M^{\Phi,\varphi}(Q)} \tag{4}$$

with $C = C(n, \Phi, \Lambda, \partial\Omega, T, \|\mathbf{a}\|_{\infty;Q}, \eta_a)$.

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On the growth of m-th derivatives of algebraic polynomials with corners in a Weighted Bergman space

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Let $G \subset \mathbb{C}$ be a finite region bounded by a Jordan curve $L := \partial G$, $\Omega := \overline{\mathbb{C}} \backslash \overline{G}$. Let $w = \Phi(z)$, $\Phi : \Omega \to \{w : |w| > 1\}$, be the univalent conformal mapping by the normalization: $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$. For $R > 1$, let us set $L_R := \{z : \ |\Phi(z)| = R\}$, $G_R := intL_R$. Let $\wp_n$ denote the class of all algebraic polynomials $P_n(z)$, $degP_n \leq n$, $n \in \mathbb{N}$.

For fixed $R_0 > 1$, let us $h : G_{R_0} \to \mathbb{R}^+$ some weight function. For arbitrary rectifiable curve L, we introduse:

$$\begin{aligned} \|P_n\|_{L_p(h,L)} \ &: = \left(\int_L h(z) \left| P_n(z) \right|^p |dz| \right)^{1/p} < \infty, \text{ if } 0 < p < \infty; \\ \|P_n\|_{L_\infty(1,L)} \ &: = \max_{z \in L} |P_n(z)|, \text{ if } p = \infty. \end{aligned}$$

In this work, we continue the study of the problem of pointwise estimates of the m-th derivatives $\left| P_n^{(m)}(z) \right|$, $m \geq 1$, in unbounded regions with a piecewise Dini-smooth boundary, having the exterior nonzero and interior zero angles and for the Jacobi weight function $h(z)$ and obtain estimates of the following type:

$$\left| P_n^{(m)}(z) \right| \leq \eta_n(L, h, p, z, m) \|P_n\|_{L_p(h,L)}, \ z \in \Omega,$$

where $\eta_n := \eta_n(G, h, p, z, m) \to \infty$ as $n \to \infty$, depending on the properties of L, h and of distance from $z \in \Omega$ to L.

Effects of layer and gas separation temperatures on gas diffusion in porous media

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Maintaining a high rate in the process of displacement in hydrocarbons leads to the emergence of many negative phenomena associated with phase conductivity. In this direction, it is important to develop technologies that can increase efficiency, which is distinguished by many positive aspects.

As we currently know, in order to obtain residual parts of hydrocarbon resources from these directions, by adding a certain amount of gas to the displacing liquid, it has been successfully changed the rheological properties of the liquid, the displace condition, and the filtration characteristics. In addition, the authors [1] investigated the kinetics of carbon dioxide in-situ generation, displacement stability, and provided important solutions.

In this work, temperature changes in the process of dynamic expansion of carbon dioxide generated in the reservoir as a result of a chemical reaction and its diffusion into the displacing liquid are investigated. Since the effect of reservoir temperature on the temperature change in the gas generation is inevitable, the change of the process was accepted as a harmonic law. The following temperature distribution occurs during the formation of gas in the reservoir:

$$T = T_0 + \theta_0 e^{-kz} \cos\left(wt - kz\right), \quad k = \sqrt{\frac{\omega}{2\chi}} \tag{1}$$

For a complete mathematical formulation of the problem only the temperature dependence of the molecular diffusion coefficient is taken into account [2]:

$$D_{mol} = \frac{k_B T}{2\pi\mu R} \cdot \frac{\mu + \nu}{2\mu + 3\nu} \tag{2}$$

Here, k_B is Bolstman's constant, μ is the dynamic viscosity of the liquid in which molecules of size "R" are distributed. The value of ν parameter determined from experimental studies [3-4] was $\nu = 5.47 \cdot 10^{-5} Pa{\cdot}$sec.

Equation (2) describes molecular diffusion without considering other effects when the mean free path corresponds to the pore size [5]. Using this formula, it is estimated how the mentioned parameters change depending on each other (Fig. 1).

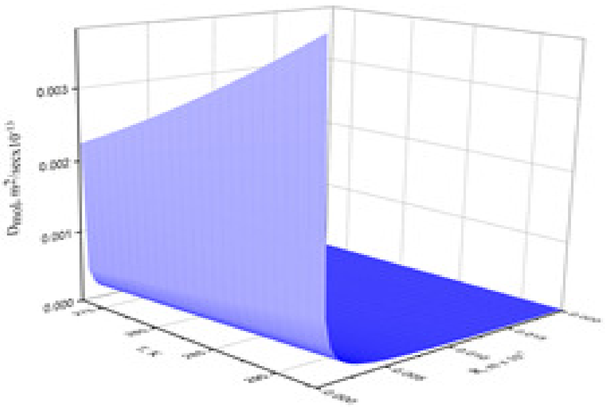

Fig. 1

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Influence of cyclic shock waves on reservoir structures and formations

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The following work theoretically evaluates the influence of the impact of the pressure of several shock waves on the change in soil porosity. During the propagation of the pressure shock wave, obviously, the soil undergoes compressive deformation to a greater extent, reservoirs are formed and to a lesser extent relaxation. As a result, the arrangement of soil particles changes relative to each other, in the opposite to the undisturbed state, and in the new state, the soil acquires changed values of density, porosity, and specific displacement of soil particles. The wave propagates into an acoustic wave at a distance and almost with no noticeable effect on the soil structure. [5] Here it would be interesting to consider approximately the region of displacements 4 - the region of weak influences. Based on the simplified soil models of Illinsky [3] and Sinitzin's generalization [2], as well as the definition of Sarimsakov's displacements [4], in the article [1] the results for the distribution of displacements and stresses during the propagation of a series of several shock waves are obtained. According to the proposed formulas:

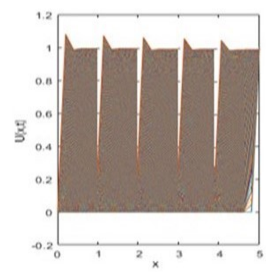
$$P_n = -(\sigma_{xx})_n = \rho_n \alpha_n^2 \left((\varepsilon)_n - (\theta_*)_n\right), (\varepsilon)_n = \frac{\partial u_n}{\partial x},$$

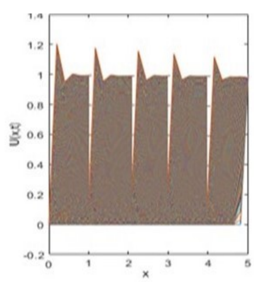
$$\alpha_n = \sqrt{\frac{P_n^*}{\rho_n \left((\varepsilon_*)_n - (\theta_*)_n\right)}}, (\varepsilon_*)_n = \frac{\rho_n - \rho_{n-1}}{\rho_n} n \geq 1.$$

where θ_* is the amount of deformation at the intersection of the straight line $P(t) = k\varepsilon$ with the axis 0 ε, $\varepsilon_* > \theta_*$. ρ_n - soil density at $\varepsilon = \varepsilon_*$. α_n - the speed of propagation of elastic waves in the pressed soil. u-displacement of the shock wave front. P^* and ε_*- critical values of pressure and deformation for a given soil.

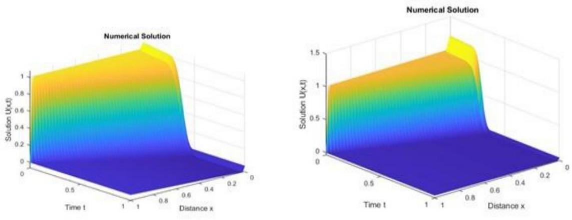
Displacement distribution graphs for dolomite and sandstone were obtained (Fig. 1).

For n-wave along the coordinate





For the 1st wave in coordinate and in time



Dolomite *Sandstone*

Fig. 1. Displacement distribution graphs.

Then it is considered relative changes in densities, porosity, and relative and specific displacements during the propagation of these waves. With the propagation of each wave with different values of the maximum pressure $P_{\max}$, the value of the pseudo-strain θ_* at the intersection of the straight line $P(t) = k\varepsilon$ with the axis $0\varepsilon - \theta_*$ (Fig. 2) remains constant.

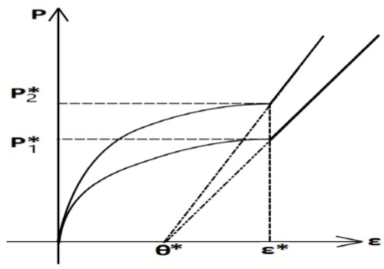


Fig. 2. Approximate diagram of compression during the propagation of two shock waves

This leads to the relative change in porosity, i.e. by what percentage does the porosity decrease in relation to the initial real value of the intrinsic porosity of the soil m, which is taken as a reference point 0. For example, for dolomites, the initial real porosity m is estimated at 14-17%, for sandstones 15-25%.

The results of the work showed that during the propagation of each wave: the density increases significantly for sandstone, and much less noticeably for dolomite (relative to sandstone). The porosity obviously also drops vastly with each next wave. For these materials this effects occur due to the higher initial density of dolomite, the initial smaller number of pores in it and the more rigid bond of the rock skeleton. Also, the maximum displacements of soil particles and, accordingly, the specific displacements and displacement increment also decrease, due to the smaller free space that remains during the compaction of the material with each subsequent wave.

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Modeling the solution to a non-stationary quasi-static problem of thermoelastic deformation of non-homogeneous bodies

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The modeling of the solution to quasi-static uncoupled problems of thermoelasticity is considered in the article by the finite element method. Forming a non-stationary problem of heat conduction, we obtain a system of differential equations in the following form [1,2]:

$$C[\dot{q}] + B[q] = P, \tag{1}$$

where $C = \sum\limits_{e=1}^{m} \rho \int\limits_{\Omega} [N^e]^T [N^e] d\Omega^e$, $B = \sum\limits_{e=1}^{m} \left(\lambda \int\limits_{\Omega} [B^e]^T [B^e] d\Omega^e + \alpha \int [N^e]^T [N^e] d \right)$,
$P = \sum\limits_{e=1}^{m} \alpha \int [N^e]^T T_\infty d^e$,

where ρ- is the density, α- heat transfer coefficient, λ - thermal conductivity coefficient, c - specific heat capacity, T_∞- ambient temperature, m â€“ the number of finite elements.

Replacing the time derivative in (1), the finite difference is:

$$\left(\frac{C}{\tau} + B \right) q^t = P + \frac{C}{\tau} q^{t-1}, \tag{2}$$

where τ is the time step. Thus, the original problem is reduced to a system of linear algebraic equations, solved by Newton's method. The thermoelasticity equation for the case when the body is subjected to surface forces and thermal loads arising in a non-uniform thermal field in tensor form has the following form:

$$\int\limits_{\Omega} (\lambda \varepsilon \delta \varepsilon + 2\mu \varepsilon_{ij} \delta \varepsilon_{ij})\, d\Omega - \int\limits_{\Omega} (3\lambda + 2\mu)\, \alpha\, (T - T_\infty)\, \delta \varepsilon d\Omega = \int \sigma_{ij} \delta u_i n_j d \tag{3}$$

where $\varepsilon = \varepsilon_{xx} + \varepsilon_{yy}$ is the volumetric deformation, α is the coefficient of thermal expansion, $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$, $\mu = \frac{E}{2(1+\nu)}$ are the Lame constants, δ_{ui} is virtual displacements, ε_{ij} are the strain tensor components i,j=1,2. n_j are the direction cosines of the outer normal to the boundary surface.

Equation (3) for one element e can be written as:

$$\int\limits_{\Omega^e} [B^e]^T [D] [B^e] d\Omega^e \{U^e\} = \int\limits_{\Omega^e} [B^e]^T [D] \{\varepsilon_T^e\} d\Omega^e + \int \{N^e\} \{P\} d \tag{4}$$

where $\{\varepsilon_T^e\} = \alpha_T \Delta T \{1\,1\,0\}^T$ is the temperature strain vector, $\{P^e\}$ is the vector of surface loads.

Summing (4) over all elements, we obtain a system of linear algebraic equations

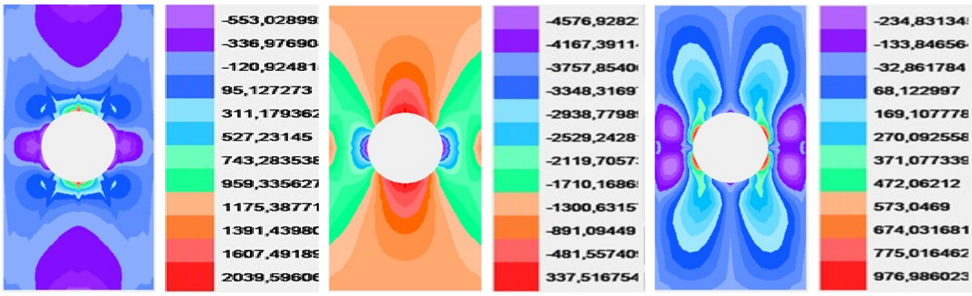

Figure 1: Values of stress component $\sigma_{xx}, \sigma_{yy}, \tau_{xy}$, $t_k = 60\,min$

$$K\{U\} = F$$

where $\{U\}$ is the vector of unknown coefficients for the displacement function, K is the stiffness matrix, F is the vector of nodal loads.

$$K = \sum_{e=1}^{m} \int_{\Omega^e} [B^e]^T [D] \{\varepsilon_T^e\} d\Omega^e + \int \{N^e\} \{P^e\} d.$$

Thus, the solution to the boundary value problem is divided into two parts: the determination of the temperature field, after which it becomes possible to determine the field of displacements and stresses of the thermoelastic medium.

The problem of compression along the Oy-axis of a copper plate with a central round hole is considered. The outer boundaries of the plate are thermally insulated. Plate dimensions are 1m x 2m, initial temperature is 20°. The radius of the hole is 0.25 m, the temperature set at the boundary is 100°. Thermophysical and mechanical parameters are: $\rho = 8890 kg/m^3$- is the density, $\alpha_T = 16.7 \cdot 10^{-6} K^{-1}$ is the heat transfer coefficient, $\lambda = 390\,/(m \cdot K)$- is the thermal conductivity coefficient, $c = 385\,J/(kg \cdot K)$ - is the specific heat capacity,$\tau = 60$ sec, $t_k = 60\,min$, $E = 128.7\,GPa$, $\mu = 0.35$, $P_{yy} = 1\,000\,Pa$.

Figure 1 shows the field of distribution of the values of the normal and tangential stress components at time $t_k = 60\,min$. Analyzing the values of the stress components Ïfxx, distribution, it can be noted that a domain of compressive values is formed along the horizontal diametral section of the hole vicinity, and the domain of maximum tensile values is formed along the vertical one. Regarding the values of σ, it should be noted that the applied compressive external loads and the temperature at the boundary of the hole lead to the appearance of the maximum concentration of compressive stresses in the vicinity of the horizontal section and in an insignificant area of tensile stresses - along the vertical section. An interesting pattern is observed when analyzing the distribution of the shear stress field τ. In some regions around the hole, the centers of which are determined by the polar angle of $\psi \approx \pm 20°$ (and symmetrical with them), the shear stresses reach a maximum value, which can lead to the deformation of the structure [3].

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On the uniform integrability of a family of moments of the first intersection of a parabola by a perturbed random walk described by an autoregressive process $(AR(1))$

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Let $\xi_n,\ n \geq 1$ be a sequence of independent identically distributed random variables defined on some probability space (Ω, F, P). It is well known that the first-order autoregression process $(AR(1))$ is defined using a recurrent relation of the form

$$X_n = \beta X_{n\ -1} + \xi_n, \quad n \geq 1,$$

where $\beta \in (-\infty, +\infty)$- is some fixed number and it is assumed that the initial value X_0 is independent of the innovation $\{\xi_n\}$.

In [1] consider, as an example, a perturbed Markov random walk described by a process $AR(1)$ of the form

$$D_n = \frac{T_n^2}{S_n}, n \geq 1$$

where $T_n = \sum\limits_{k=1}^{n} X_k X_{k-1}$ Â¸ $S_n = \sum\limits_{k=1}^{n} X_{k-1}^2$, $n \geq 1$.

In [1] others consider a family of first passage times of the form

$$t_a = \inf\{n \geq 1 :\ D_n \geq a\}$$

and the limiting distribution of overshoot $K_a =\ D_{t_a} - a$ was found as $a \to \infty$.

In [2] consider the family of first passage times of the form

$$\tau_a = inf\left\{n \geq 1 :\ D_n \geq a\sqrt{n}\right\},$$

where $a \geq 0$.

Let conditions $E\xi_1 = 0$, $D\xi_1 = 1$, $EX_0^2 < \infty$ and $0 < |\beta| < 1$ hold. Under these conditions the following results hold (see[2])

$$P\left(\sqrt{n}\left(\frac{D_n}{n} - \lambda\right) \leq x\right) = \Phi\left(\frac{x}{\alpha}\right),$$

$$x \in R, \quad \alpha = \frac{|\beta|}{\sqrt{1-\beta^2}}, \quad \lambda_0 = \frac{\beta^2}{1-\beta^2},$$

$$\frac{\tau_a}{N_a}\ a.s. \quad \to\ 1\ as\ a \to \infty,$$

where $N_a = (a/\lambda)^2$ and

$$P\left(\sqrt{N_a}\left(\frac{\tau_a}{N_a}-1\right)\leq x\right)=\Phi\left(\frac{\alpha}{2}x\right) \ .$$

Let $\delta(n) = sup_x \left|P\left(\sqrt{n}\left(\frac{D_n}{n}-\lambda\right)\leq x\right)-\Phi\left(\frac{x}{\alpha}\right)\right|$.

First main results of the present work we can formulate of the form:

Theorem. Suppose that $E\xi_1 = 0$, $E{\xi_1}^2 = 1$, $EX_0^2 < \infty$ and $0 < |\beta| < 1$. Moreover, assume that, $\sum\limits_{n=1}^{\infty}\delta(n) < \infty$. Then the family $\frac{\tau_a}{a^2}$, $a \geq a_0$ is uniformly integrable, where $a_0 > 0$ is some fixed number.

Corollary. Suppose that the conditions of Theorem be satisfied. Then

$$\frac{E\tau_a}{N_a}\to 1, a\to\infty.$$

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The Lemb problem for an elastic system layer and half-plane

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A complicated version of the Lemb problem is considered, when the region consists of an elastic layer and a half-plane, from different materials. The existence of an inhomogeneous medium greatly aggravates the process of obtaining an analytical solution. The problem is solved using the integral Laplace and Fourier transformations, as in the original form of the Lemb problem. However, a completely new technique is used to find the originals of received transformations. This technique has previously been successfully used in solving other problems of non-stationary dynamics of elastic bodies, the author of which is one of the authors of this paper. Analytical explicit solutions are obtained for the early stages of the process.

Weakly periodic p-adic generalized Gibbs measures for the Ising model on the Cayley tree of order two

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In the present paper, we study the $p-$adic Ising model on the Cayley tree of order two. The existence of H_A-weakly periodic (non-periodic) $p-$adic generalized Gibbs measures for this model is shown.

Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime p, every rational number $x \neq 0$ can be represented in the form $x = p^r \frac{n}{m}$, where $r, n \in \mathbb{Z}$, m is a positive integer, and m and n are relatively prime with p, r is called the order of x and written $r = ord_p x$. The $p-$adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

This norm is non-Archimedean and satisfies the so-called strong triangle inequality $|x+y|_p \leq \max\{|x|_p, |y|_p\}$ for all $x, y \in \mathbb{Q}$.

The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field $\mathbb{Q}_p$ (see [1]).

The completion of the field of rational numbers $\mathbb{Q}$ is either the field of real numbers $\mathbb{R}$ or one of the fields of p-adic numbers $\mathbb{Q}_p$ (Ostrowski's theorem).

Any p-adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...)$$

where $\gamma(x) \in \mathbb{Z}$ and the integers x_j satisfy: $x_0 \neq 0,\ x_j \in \{0, 1, ..., p-1\}$, $j \in \mathbb{N}$ (see [1]). In this case $|x|_p = p^{-\gamma(x)}$.

Let us consider the following set on the Cayley tree $\Gamma^k(V, L)$ (see [2]). Let $x_o \in V$ be fixed, $W_n = \{x \in V : |x| = n\}$, $V_n = \{x \in V : |x| \leq n\}$, $L_n = \{l = \langle x, y\rangle \in L : x, y \in V_n\}$, $S(x) = \{y \in V : x \to y\}$, $S_1(x) = \{y \in V : d(x, y) = 1\}$. And for $x \in W_n$, denote $S(x) = \{y \in W_{n+1} : d(x, y) = 1\}$. The set is called direct successors of x. The set $S(x)$ is called the set of direct successors of the vertex x.

We consider p-adic Ising model on the Cayley tree Γ^k. Let $\mathbb{Q}_p$ be a field of p-adic numbers and $\Phi = \{-1, 1\}$. A configuration σ on V is defined by the function $x \in V \to \sigma(x) \in \Phi$. Similarly, one can define the configuration σ_n and $\sigma^{(n)}$ on V_n and W_n, respectively. The set of all configurations on V (resp. V_n, W_n) is denoted by $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$).

For given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\varphi^{(n)} \in \Omega_{W_n}$ we define a configuration in Ω_{V_n} as follows

$$(\sigma_{n-1} \vee \varphi^{(n)})(x) = \begin{cases} \sigma_{n-1}(x), & \text{if } x \in V_{n-1}, \\ \varphi^{(n)}(x), & \text{if } x \in W_n. \end{cases}$$

A formal p-adic Hamiltonian $H : \Omega \to \mathbb{Q}_p$ of the p-adic Ising model is defined by

$$H(\sigma) = J \sum_{<x,y>\in L} \sigma(x)\sigma(y),$$

where $0 < |J|_p < p^{-1/(p-1)}$ for any $\langle x, y\rangle \in L$.

We define a function $h : x \to h_x$, $\forall x \in V \setminus \{x_0\}$, $h_x \in \mathbb{Q}_p$ and consider p-adic probability distribution $\mu_h^{(n)}$ on Ω_{V_n} defined by

$$\mu_h^{(n)}(\sigma_n) = \frac{1}{Z_n^{(h)}} \exp_p\{H_n(\sigma_n)\} \prod_{x\in W_n} h_{\sigma(x),x} \quad n = 1, 2, \dots,$$

where $Z_n^{(h)}$ is the normalizing constant

$$Z_n^{(h)} = \sum_{\varphi\in\Omega_{V_n}} \exp_p\{H_n(\varphi)\} \prod_{x\in W_n} h_{\sigma(x),x}.$$

A p-adic probability distribution $\mu_h^{(n)}$ is said to be consistent if for all $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, we have

$$\sum_{\varphi\in\Omega_{W_n}} \mu_h^{(n)}(\sigma_{n-1} \vee \varphi) = \mu_h^{(n-1)}(\sigma_{n-1}).$$

In this case, by the p-adic analogue of the Kolmogorov theorem there exists a unique measure μ_h on the set Ω such that $\mu_h\left(\{\sigma|_{V_n} \equiv \sigma_n\}\right) = \mu_h^{(n)}\ (\sigma_n)$ for all n and $\sigma_n \in \Omega_{V_n}$.(see [3])

Theorem 1. *If $p \equiv 1(\mathrm{mod}\, 4)$ then there exists at least two weakly periodic (non-periodic) p-adic generalized Gibbs measures for the Ising model on the Cayley tree of order two.*

Remark. *In [4] it was proved that for the Ising model on a Cayley tree of order $k = 2$ with respect to the normal divisor of index 2, there does not exist a weakly periodic (non-translation-invariant) Gibbs measure in the real case. In the p-adic case Theorem 1 was shown that for the Ising model, there are at least two new weakly periodic p-adic generalized Gibbs measures.*

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On eigen-values and eigen-functions of a Fredholm type integral operator

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On the space $C[-\pi, \pi]$ we consider the integral operator

$$Ax = \int_{-\pi}^{\pi} \sin(t+s)x(s)ds.$$

By the definition we must find such values of the parameter λ that

$$\int_{-\pi}^{\pi} \sin(t+s)x(s)ds = \lambda x(t), -\pi \leq t \leq \pi$$

be the solution of the equation $x(t) \neq 0$.

Writing in the equation $\sin(t+s) = \sin t \cos s + \cos t \sin s$ the equation will be in the form

$$\lambda x(t) = \sin t \int_{-\pi}^{\pi} \cos sx(s) + \cos t \int_{-\pi}^{\pi} \sin sx(s)ds, -\pi \leq t \leq \pi$$

It is seen from this equality that the solution of the equation should be sought in the form of $x(t) = \alpha \sin t + \beta \cos t$. Substituting this expression in the above equation , we obtain:

$$\alpha\lambda \sin t + \beta\lambda \cos t = \beta\pi \sin t + \alpha\pi \cos t.$$

Since the functions *sint* and *cost* are linear independent on the intervals $[-\pi, \pi]$, we obtain

$$\begin{cases} \alpha\lambda - \beta\pi = 0 \\ \alpha\pi - \beta\lambda = 0 \end{cases}$$

Since according to the condition $x(t) \neq 0$, even if one of α or β should be non-zero . Therefore, $\Delta = \begin{vmatrix} \lambda & \pi \\ \pi & \lambda \end{vmatrix} = 0, \lambda^2 - \pi^2 = 0, \lambda_1 = -\pi, \lambda_2 = \pi.$

For $\lambda_1 = -\pi$ for $\alpha = -\beta$, for $\lambda_2 = -\pi$ $\alpha = \beta$. This means that the eigen value $\lambda_1 = -\pi$ corresponds to the eigen function $x(t) = \alpha(\sin t - \cos t)$, the eigen value $\lambda_2 = \pi$ corresponds to the eigen-function $x(t) = \alpha(\sin t + \cos t)$.

Let us show that the number $\lambda = 0$ is also an eigen -value of the operator. For that we should show the existence of non-zero solution of the equality $\int_{-\pi}^{\pi} \sin(t+s)x(s)ds = 0$

i.e. there should be such $x(t) \neq 0$ that

$$\sin t \int_{-\pi}^{\pi} \cos sx(s)ds + \cos t \int_{-\pi}^{\pi} \sin sx(s)ds = 0.$$

Since sin t and cost are linear independent, form this equality we obtain

$$\int_{-\pi}^{\pi} \cos sx(s)ds = 0, \int_{-\pi}^{\pi} \sin sx(s)ds = 0$$

These equalities show that all x(t) functions that are orthogonal to the functios cos t and sin t in the space $L_2(-\pi, \pi)$ are eigen-functions of the operator. For example, the functions $x(t) = const, \quad n = 0, 2, 3..., x(t) = \sin t, n = 0, 2, 3, ...$ are eigen-functions. In general, we can show that there exist infinitely many eigen-functions corresponding to the eigen-value $\lambda_3 = 0$. All continuous functions that are orthogonal to the functions sin t and cos t on the interval $[-\pi, \pi]$ are eigen-values of the operator.

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Some Zygmund type estimates for a singular integral

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Let R^nbe n-dimensional Euclidean space of points $x = (x_1, x_2, ..., x_n)$, $B(a,r) :=$ $\{x \in R^n \colon |x-a| \leq r\}$ be a closed sphere in R^n of radius $r > 0$ centered at the point $a \in R^n$, Nbe the set of all natural numbers $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $x^\nu = x_1^{\nu_1} \cdot x_2^{\nu_2} \cdots x_n^{\nu_n}$, $|\nu| = \nu_1 + \nu_2 + ... + \nu_n$, where $\nu_1, \nu_2, ..., \nu_n$ are non-negative integers. By $L_{loc}(R^n)$ we denote the totality of all locally summable in R^n functions.

Let $f \in L_{loc}(R^n)$, $k \in \mathrm{N} \bigcup \{0\}$. We consider the polynomial (see [1], [2])

$$P_{k,\mathrm{B}(a.r)} f(x) :=$$

$$= \sum_{|\nu| \leq k} \left(\frac{1}{|\mathrm{B}(a,r)|} \int_{\mathrm{B}(a,r)} f(t) \varphi_\nu \left(\frac{t-a}{r} \right) dt \right) \varphi_\nu \left(\frac{x-a}{r} \right),$$

where $|B(a,r)|$ denotes the volume of the sphere $B(a,r)$ and $\{\varphi_\nu\}$, $|\nu| \leq k$, is an orthonormed system obtained as a result of application of orthogonalization process with respect to the scalar product

$$(f,g) := \frac{1}{|B(0,1)|} \int_{B(0,1)} f(t) g(t) dt$$

to the system of power functions $\{x^\nu\}$, $|\nu| \leq k$, located in partially lexicographic order [3]. $P_{k,\, B(a,r)} f$ is a polynomial of degree not higher than k. We denote the totality of all polynomials in R^n of degree not higher than k by P_k. Thus, $P_{k,\, B(a,r)} f \in P_k$.

For the function $f \in L_{loc}^p(R^n)$ $(1 \leq p \leq \infty)$ we define the following functions:

$$\Omega_k(f, B(a,r))_p :=$$

$$= \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} \left| f(t) - P_{k-1,B(a,r)} f(t) \right|^p dt \right)^{1/p}, (1 \leq p < \infty),$$

$$\Omega_k(f, B(a,r))_\infty := ess \sup \left\{ \left| f(t) - P_{k-1,B(a,r)} f(t) \right| : \ t \in B(a,r) \right\},$$

$$m_f^k(x;\delta)_p := \sup \left\{ \Omega_k f(B(x,r))_p : \ \ r \leq \delta \right\} (x \in R^n, \ \ \delta > 0),$$

Let $1 \leq p, q \leq \infty$. We introduce the function

$$m_f^k(r)_{pq} := \begin{cases} \left\| m_f^k(\cdot\, ; r)_p \right\|_{L^q(R^n)}, & for \ \ 1 \leq q < \infty, \\ \sup\limits_{x \in R^n} m_f^k(x;r)_p, & for \ \ q = \infty. \end{cases}$$

We consider a singular integral operator (see, e.i. [2], [4]):

$$A_k f(x) = \lim_{\varepsilon \to +0} \int_{R^n} \left\{ K_\varepsilon(x-y) - \left(\sum_{|\nu| \le k-1} \frac{x^v}{v!} D^v K(-y) \right) X_{\{|t|>1\}}(y) \right\} f(y)dy,$$

where

$$K(x) = \omega(x) \cdot |x|^{-n}, \int_{S^{n-1}} \omega(x)ds = 0, K_\varepsilon(x) = K(x) \cdot X_{\{|t|>\varepsilon\}}(x),$$

The function $\omega(x)$ is homogeneous and is of degree 0, $X_{\{|t|>\varepsilon\}}$ is a characteristic function of the set $\{t \in R^n : \ |t| > \varepsilon\}$, S^{n-1} is a unit sphere in the Euclidean space R^n; We assume that for $k = 1$ the function $K(x)$ is differentiable and has bounded partial derivatives of first order, and for $k > 1$ the function $K(x)$ is k-times continuously differentiable on the sphere S^{n-1}; $v = (v_1, v_2, ...v_n)$, $v_1, v_2, ..., v_n$ are entire non-negative numbers, $|v| = v_1 + v_2 + ... + v_n$, $v! = v_1!v_2!...v_n!$ $k \in N$,

$$D^v f := \frac{\partial^{|v|} f}{\partial x_1^{v_1} \partial x_2^{v_2} ... \partial x_{1n}^{v_n}}.$$

Theorem. *Let $f \in L^p_{loc}(R^n)$, $1 < p < \infty$, $1 \le q \le \infty$. Then, when the integral in the right-hand side converges, the following inequality is valid*

$$m_{A_k f}(\delta)_{pq} \le C\delta^k \int_\delta^\infty \frac{m_f(t)_{pq}}{t^{k+1}} dt, (\delta > 0),$$

where the constant $C > 0$ is independent of f and δ.

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Some inequalities for metric characteristics

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Let R^nbe n-dimensional Euclidean space of points $x = (x_1, x_2, ..., x_n)$, $B\,(a, r) : = \{x \in R^n : \, |x - a| \leq r\}$ be a closed sphere in R^n of radius $r > 0$ centered at the points $a \in R^n$, N be the set of all natural numbers, $\nu = (\nu_1, \nu_2, ..., \nu_n)$, $x^\nu = x_1^{\nu_1} \cdot x_2^{\nu_2} \cdots x_n^{\nu_n}$, $|\nu| = \nu_1 + \nu_2 + ... + \nu_n$, where $\nu_1, \nu_2, ..., \nu_n$ are non-negative integers. We denote the totality of all functions, whose, p-th degree of modulus is locally summable in R^n by $L^p_{loc}\,(R^n)$ $(1 \leq p < \infty)$, and the totality of all locally bounded in R^n functions denote by $L^\infty_{loc}\,(R^n)$.

Let $f \in L^1_{loc}\,(R^n)$, $k \in N \bigcup \{0\}$. Let us consider the polynomial (see [1], [3])

$$P_{k,\mathrm{B}(a.r)} f\,(x) := \sum_{|\nu| \leq k} \left(\frac{1}{|\mathrm{B}\,(a, r)|} \int_{\mathrm{B}(a,r)} f\,(t)\, \varphi_\nu \left(\frac{t - a}{r} \right) dt \right) \varphi_\nu \left(\frac{x - a}{r} \right),$$

where $|B\,(a, r)|$ denotes the volume of the sphere $B(a, r)$ and $\{\varphi_\nu\}$, $|\nu| \leq k$, is an orthonormed system obtained as a result of the application of the orthogonalization process with respect to the scalar product

$$(f, g) := \frac{1}{|B(0, 1)|} \int_{B(0,1)} f(t) g(t) dt$$

to the system of power functions $\{x^\nu\}$, $|\nu| \leq k$, located in partial lexicographic order [4].

$P_{k,\, B(a,r)} f$ is a polynomial of degree not higher than k. We denote the totality of all polynomials in R^n of degree not higher than k by P_k. Thus, $P_{k,\, B(a,r)} f \in P_k$.

Let $f \in L^p_{loc}\,(R^n)$, $1 \leq p \leq \infty$, $k \in N$. We consider the function

$$M^k_f\,(x; r)_p := \inf_{\pi \in P_{k-1}} |B\,(x, r)|^{-\frac{1}{p}}\, \|f - \pi\|_{L^p(B(x,r))}\, (r > 0, \;\; x \in R^n)\,.$$

It is easy to verify that

$$\exists C > 0 \forall x \in R^n \forall r > 0 :$$

$$M^k_f\,(x; r)_p \leq \Omega_k\,(f, B\,(x, r))_p \leq C \cdot M^k_f\,(x; r)_p\,,$$

where

$$\Omega_k\,(f, B\,(x, r))_p := |B\,(x, r)|^{-\frac{1}{p}} \left\| f - P_{k-1,B(a,r)} f \right\|_{L^p(B(a,r))}.$$

The quantity $\Omega_k\,(f, B\,(x, r))_p$ is called k-th order mean oscillation of the function f in the sphere $B\,(x, r)$ in the metrics L^p.

Let $1 \leq p, q \leq \infty$. We introduce the function

$$M_f^k(r)_{pq} := \begin{cases} \left\| M_f^k(\cdot\,; r)_p \right\|_{L^q(R^n)}, & for\ \ 1 \leq q < \infty, \\ \sup\limits_{x \in R^n} M_f^k(x; r)_p, & for\ \ q = \infty. \end{cases}$$

We can verify that

$$f \in BMO(R^n) \quad \Leftrightarrow \quad \left(f \in L^1_{loc}(R^n),\ \ M_f^1(\delta)_{1\infty} = O(1) \qquad (\delta > 0)\right),$$

$$f \in BMO_\varphi \quad \Leftrightarrow \quad \left(f \in L^1_{loc}(R^n),\ \ M_f^1(\delta)_{1\infty} = O(\varphi(\delta)) \qquad (\delta > 0)\right).$$

To determine the spaces $BMO(R^n)$ and $f \in BMO_\varphi$ see, e.i. [2].

The k-th order continuity modulus ($k \in N$) of the function f in the metrics L^p ($1 \leq p \leq \infty$) is determined by the equality

$$\omega_f^k(r)_p := \sup\left\{ \left\|\Delta_h^k f\right\|_{L^p(R^n)} : \quad |h| \leq r \right\}, (r > 0),$$

where $\Delta_h^1 f(x) := f(x+h) - f(x)$, $\Delta_h^k f = \Delta_h^1\left(\Delta_h^{k-1} f\right)$.

Note that we will call two functions equivalent if they coincide almost everywhere.

Theorem 1. *If $f \in L^p(R^n)$, $(1 \leq q \leq p \leq \infty)$ (for $p = \infty$ it is assumed that f is equivalent to the continuous function), we have the inequality*

$$M_f^k(\delta)_{qp} \leq C \cdot \omega_f^k(\delta)_p\,(\delta > 0), \tag{1}$$

where the constant $C > 0$ is independent of f and δ.

Theorem 2. *Let $f \in L^q_{loc}(R^n)$, $1 \leq p \leq \infty$, $1 \leq q < \infty$ and*

$$\int_0^1 \frac{M_f^k(t)_{qp}}{t}\,dt < +\infty.$$

then the following inequality is valid

$$\omega_f^k(\delta)_p \leq C \cdot \int_0^\delta \frac{M_f^k(t)_{qp}}{t}\,dt\ \ (\delta > 0), \tag{2}$$

where the constant $C > 0$ is independent of f and δ. Furthermore, if $p = \infty$, thenf is equivalent to the continuous function.

Theorem 3. *If $f \in L^\infty_{loc}(R^n)$, $k \in N$, the following inequality is valid*

$$\omega_f^k(\delta)_\infty \leq C \cdot M_f^k(\delta)_{\infty\infty}\,(\delta > 0),$$

where $C > 0$ is independent of f and δ.

Theorem 4. *Let $f \in L^\infty(R^n)$ and f be equivalent to the continuous function. Then we have the following inequality*

$$C_1 \cdot M_f^k(\delta)_{\infty\infty} \leq \omega_f^k(\delta)_\infty \leq C_2 \cdot M_f^k(\delta)_{\infty\infty}\,(\delta > 0),$$

where positive constants C_1 and C_2 are independent of f and δ.

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Free vibrations of an inhomogeneous fluid-contacting shell strengthened with inhomogeneous rods

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In the present paper the study free vibrations of an inhomogeneous fluid-contacting shell strengthened with rods. The Hamilton-Ostogradsky variational principle was used when solving the problem . It was accepted that the nonhomogeneity of rods used in the strengthening change by the exponential law. The nonhomogenity of the cylindrical shell change by the linear law in the direction of the thickness. The fluid was accepted as ideal. Rigid contact condition between the rods and the cylindrical shell was considered. Using the contact conditions, the frequency equation was structured, the roots were found implemented by the numerical method, characteristical curves were built. In [1], this problem was executed for a nonhomogeneous cylindrical shell strengthened with inhomogeneous rings.

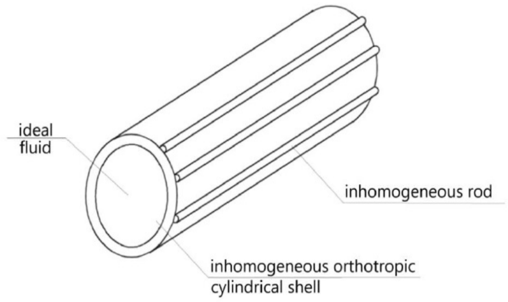

Fig 1. Inhomogeneous, liquid-contacting cylindrical shell strengthened with inhomogeneous rods.

Assume that an inhomogeneous orthotropic cylindrical shell inhomogeneous in thickness and located in ideal fluid was strengthened with inhomogeneous rods.(fig 1) Since the represented construction consists of fluid, a rod and cylindrical shell, its total energy is in the form

$$J = V + V_m + A_m \tag{1}$$

where V is the total energy of an orthotropic cylindrical shell inhomogeneous in thickness and its expression was given in [1] ,V_m is the total energy of inhomogeneous rods. A_m is the work done by the forces acting an the cylindrical shell as viewed from fluid at the displacements of the points of the shell.

Under the cylindrical shell strengthened with rods we understand a cylindrical shell and a system consisting of rods rigidly strengthened to it along the coordinate lines. It is considered that the coordinate axes coincide with the principal curvature lines of the cylindrical shell and are in rigid contact along these lines. So, the following conditions between the cylindrical shell and rods are satisfied:

$$u_i\left(x\right) = u\left(x, y_i\right) + h_i\varphi_1\left(x, y_i\right),\ \ \vartheta_i\left(x\right) = \vartheta\left(x, y_i\right) +$$

$$+h_i\varphi_2\left(x, y_i\right) w_i\left(x\right) = w\left(x, y_i\right),\ \ \varphi_i\left(x\right) = \varphi_1\left(x, y_i\right),\ w_i\left(x\right) = w\left(x, y_i\right),$$

$$\varphi_i\left(x\right) = \varphi_1\left(x, y_i\right),\ \ \varphi_{kpi}\left(x\right) = \varphi_2\left(x, y_i\right);\ \ h_i = 0,5h + H_i^1 \tag{2}$$

The following contact conditions between the fluid and cylindrical shell are satisfied.

$$\vartheta_r|_{r=R} = -\left(\omega\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right) \tag{3}$$

$$q_z = -p_{|r=R} \tag{4}$$

So, the solution of the problem of vibrations of a cylindrical shell dynamically contacting with fluid and strengthened with rods is reduced to joint integration of the of total energy (1) of the construction consisting of a cylindrical shell with flowing fluid in the inner area and strengthened with discretely distributed inhomogeneous rods under the contact conditions (2)-(4)

$$\begin{aligned} u &= u_0\cos\chi\xi\cos n\theta\sin\omega_1 t_1 \\ \vartheta &= \vartheta_0\sin\chi\xi\sin n\theta\sin\omega_1 t_1 \\ w &= w_0\sin\chi\xi\cos n\theta\sin\omega_1 t_1 \end{aligned} \tag{5}$$

here u_0, ϑ_0, w_0 are unknown constants,

$$\omega_1 = \sqrt{\frac{\left(1-v^2\right)\rho_0 R^2\omega^2}{E}} = \frac{\omega}{\omega_0}\omega_0 = \sqrt{\frac{E}{\left(1-v^2\right)\rho_0 R^2}},\ \xi = \frac{x}{L}, t_1 = \omega_0 t,\ \ \chi, n$$

is the generatrix and wave numbers of the cylindrical shell in the circular direction. Substituting the solutions (5) in (1), taking into account contact conditions (2)-(4), we get an expression with respect to the unknown constants u_0, ϑ_0, w_0 . The obtained expression will be a second degree polynomial with respect to the constants u_0, ϑ_0, w_0

$$J_m = \varphi_{11}u_0^2 + \varphi_{22}\vartheta_0^2 + \varphi_{33}w_0^2 + \varphi_{44}u_0\vartheta_0 + \varphi_{55}u_0 w_0 + \varphi_{66}\vartheta_0 w_0 \tag{6}$$

Since we expression $\varphi_{ii}(i = 1, 2, 3, \ldots, 6)$ are bulky , we do not give them here. Varying this expression with respect to the independent constants u_0, ϑ_0, w_0 and equating to zero the coefficients of independent variations, we get a system of inhomogeneous algebraic equations. Since the obtained system is a system of homogeneous linear algebraic equations, the necessary and sufficient condition for the existence of its nontrival solution is the equality of its principal determinant to zero. As a result, we obtain the following frequency equation.

$$4\varphi_{11}\varphi_{22}\varphi_{33} + \varphi_{44}\varphi_{55}\varphi_{66} - \varphi_{55}^2\varphi_{22} - \varphi_{66}^2\varphi_{11} - \varphi_{44}^2\varphi_{33} = 0 \tag{7}$$

The roots by the equation (7) was calculated by the numerical method. İn the calculation , the following values for the parameters characterizing the fluid , shell and rods were taken:

$$I_{kpi} = 0,23mm^4; I_{xi} = 5,1mm^4; \ \frac{l}{R} = 3,$$

$$\rho_0 = \rho_{i0} = 1850kg/m^3 \rho_m/\rho_0 = 0,15, \ g(x) = \left(1 + \mu e^{\frac{x}{l}}\right).$$

$$\bar{E}_{i0} = 6,67 \cdot 10^9 N/m^2; I_{zi} = 1,3mm^4, \ \ \nu = 0,35; \ m = 1 \ n = 8;$$

$$h_i = 1,39cm, \ F_i = 5,2 \ mm^2; A = 0,034; \ \ \psi = 0,05; \ \ f\left(z\right) = 1 + \sigma\frac{z}{h}; \ \sigma \in [0;1]$$

The results of calculations show that increasing the amount of rods, vibrations of the system at first increase, and after certain value begin to decrease. This is explained by the fact that increasing the amount of rods, its mass increases and their inertia action to the vibration amplifies. Increasing the value of the inhomogeneous parameter, natural vibrations of the system increase.

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On the asymptotics of the solution of an boundary value problem for a arbitrary odd order singularly perturbed differential equation

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In this paper, in $P = \{(t,x)|0 \le t \le 1, -\infty < x < \infty\}$ we consider the following boundary value problem

$$L_\varepsilon u \equiv (-1)^m \varepsilon^{2m} \frac{\partial^{2m+1} u}{\partial t^{2m+1}} + \varepsilon^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + au = f(t,x), \tag{1}$$

$$u|_{t=0} = \left.\frac{\partial u}{\partial t}\right|_{t=0} = ... = \left.\frac{\partial^m u}{\partial t^m}\right|_{t=0}, \tag{2}$$

$$\left.\frac{\partial^{m+1} u}{\partial t^{m+1}}\right|_{t=1} = \left.\frac{\partial^{m+2} u}{\partial t^{m+2}}\right|_{t=1} = ... = \left.\frac{\partial^{2m} u}{\partial t^{2m}}\right|_{t=1} = 0, \tag{3}$$

$$\lim_{|x|\to+\infty} u = 0, \qquad \lim_{|x|\to+\infty} \frac{\partial u}{\partial x} = 0. \tag{4}$$

Here $\varepsilon > 0$ is a small parameter, m is an arbitrary natural number, $a > 0$ is a constant, $f(x,u)$ is a given function .

In the first iterative process, the approximate solution of the equation (1) is sought in the form $W = \sum\limits_{i=0}^{n} \varepsilon^i W_i$. For the functions W_i we obtain the following recurrently connected equations

$$\frac{\partial W_0}{\partial t} + \frac{\partial^2 W_0}{\partial x^2} + aW_0 = f(t,x), \tag{5}$$

$$\frac{\partial W_k}{\partial t} + \frac{\partial^2 W_k}{\partial x^2} + aW_k = f_k(t,x), k = 1,2,...,n. \tag{6}$$

The functions $f_k(t,x)$ are known functions dependent on the functions $W_0, W_1, ..., k-1$. For equations (5)-(6) from conditions (2) with respect to t one condition will be taken. Therefore, it is necessary to construct boundary lager functions near the boundaries $t = 0$ and $t = 1$. We take change of variable $t = \varepsilon\xi$ near $t = 0$ and $1 - t = \varepsilon y$ near $t = 1$ and write new decomposition of the operator L_ε near these boundaries.

If we look for a boundary layer function near the boundary $t = 0$ in the form of $\eta = \sum\limits_{j=0}^{n+m-1} \varepsilon^{1+j} \eta_j$, a boundary layer function near the boundary $t = 1$ in the form of $\psi =$

$\sum\limits_{j=0}^{n+m-1} \varepsilon^{m+1+j}\psi_j$, then for determining the functions $\eta_j, \psi_j; j = 0, 1, ..., n+m-1$ we obtain the following equations

$$(-1)^m \frac{\partial^{2m+1}\eta_0}{\partial\xi^{2m+1}} + \frac{\partial\eta_0}{\partial\xi} = 0, \tag{7}$$

$$(-1)^m \frac{\partial^{2m+1}\eta_j}{\partial\xi^{2m+1}} + \frac{\partial\eta_j}{\partial\xi} = h_j;\ j = 1, 2, ..., n+m+1, \tag{8}$$

$$(-1)^m \frac{\partial^{2m+1}\psi_0}{\partial y^{2m+1}} + \frac{\partial\psi_0}{\partial y} = 0. \tag{9}$$

$$(-1)^m \frac{\partial^{2m+1}\psi_j}{\partial y^{2m+1}} + \frac{\partial\psi_j}{\partial y} = g_j; .j = 1, 2, ...n+m-1. \tag{10}$$

Here h_j are known functions dependent on $\eta_0, \eta_1, ..., \eta_{j-1}, g_j$ are known functions dependent on $\psi_0, \psi_1, ..., \psi_{j-1}$. Initial conditions for equations (5),(6) are found from the equality $(W+\eta)|_{t=0} = 0$ and have the form

$$W_0|_{t=0} = 0,\ W_i|_{t=1} = -\eta_{i-1}|_{\xi=0}\, 0; i = 1, 2, ...n. \tag{11}$$

Boundary conditions for equations (7),(8) can be given by the following general formula:

$$\left.\frac{\partial^k\eta_j}{\partial\xi^k}\right|_{\xi=0} = -\left.\frac{\partial^k W_{j+1+k}}{\partial t^k}\right|_{t=0};\quad k = 1, 2, .., m; j = 0, 1, ..., n+m-1 \tag{12}$$

and the functions W_r for $r < 0$ or $r > n$ should be considered as identical zeros.

Boundary conditions for equations (9),(10) are found from the requirement

$$\left.\frac{\partial^{m+k}}{\partial t^{m+k}}(W+\eta+\psi)\right|_{t=1} = 0, k = 1, 2, ..., m. \tag{13}$$

From (5),(11) and (4) we have W_0 is the solution of equation (5) satisfying the following boundary conditions

$$W|_{t=0} = 0,\ \lim_{|x|\to\infty} W_0 = 0 \tag{14}$$

We prove the following statement.

Lemma 1. *Let $f(x,t)$ be a function given in P and having continuous derivatives with respect to t to the $(n+2m+2)$-th order inclusively, infinitely differentiable with respect to x and satisfy the condition*

$$\sup_x \left(1+|x|^l\right)\left|\frac{\partial^r f(t,x)}{\partial t^{r_1}\partial t^{r_2}}\right| \le C^{(1)}_{lr_1r_2} < +\infty, \tag{15}$$

here l is a non-negative number, $r = r_1 + r_2, r_1 \le q,\ r_2$ is arbitrary, $C^{(1)}_{lr_1r_2} > 0$ is a number. Then the function $W_0(t,x)$ being the solution of problem (5), (14), satisfies the condition

$$\sup_x \left(1+|x|^l\right)\left|\frac{\partial^r W_0(t,x)}{\partial t^{r_1}\partial t^{r_2}}\right| \le C^{(2)}_{lr_1r_2} < +\infty,$$

where $r_1 \leq q+1, \quad C^{(2)}_{lr_1r_2} > 0.$

Continuing the process, we build the functions $W_i; i=0,1,...,n; \eta_j; j=0,1,...,n+m-1$. The functions ψ are built in the same way. Thus, we obtain the following representation of the solutions of problem (1)-(4) in the form

$$u = \sum_{i=0}^{n} \varepsilon^i W_i + \sum_{j=0}^{n+m-1} \varepsilon^{1+s} \eta_j + \sum_{j=0}^{n+m-1} \varepsilon^{1+m+j} \psi_j + \varepsilon^{n+1} z, \tag{16}$$

where $\varepsilon^{n+1} z$ is a remainder.

Lemma 2. *The following estimation is valid for the function* z

$$\varepsilon^m \left\| \left. \frac{\partial^m z}{\partial t^m} \right|_{t=1} \right\|_{L_2(-\infty,+\infty)} + \varepsilon^m \left\| \frac{\partial^2 z}{\partial x^2} \right\|_{L_2(P)} +$$

$$+ \left\| \left. z \right|_{t=1} \right\|_{L_2(-\infty,+\infty)} + \left\| \frac{\partial z}{\partial x} \right\|_{L_2(D)} + c_1 \left\| z \right\|_{L_2(P)} \leq c_2, \tag{17}$$

where the constants $c_1 > 0, c_2 > 0$ *are independent of* ε.

Theorem. *Let the function* $f(t,x)$ *satisfy the condition of lemma 1. Then the solution of the boundary value problem (1)-(4) is representable in the form (16), is a remainder, and estimation (17) is valid for the function* z.

On μ-strong Cesaro summability at infinity

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The concept of μ-strong Cesaro summability at infinity for a locally integrable function is introduced in this work. The concept of μ-statistical convergence at infinity is also considered and the relationship between these two concepts is established.

We will use the standard notation. R is the set of all real numbers; $\exists$ will mean "there exist(s)"; $\exists!$ will mean "there exists a unique"; $\Rightarrow$ will mean "it follows"; $\Leftrightarrow$ will mean equivalence. $I_a^{+\infty} \equiv [a\,;+\infty)$; $I_a^{-\infty} \equiv (-\infty; a]$.

Let $(I_a^{\infty}; B; \mu)$ be a measurable space with measure $\mu : B \to I_a^{\infty}$, where B σ-algebra of Borel subsets in I_a^{∞}. We will assume that the measure μ is a σ- finite measure and $\mu\,(I_a^{\infty}) = +\infty$. The measure of the set $M \in B$ will be denoted by $|M|$, i.e. $|M| = \mu\,(M)$.

Let $f : I_a^{\infty} \to R$ be some B- measurable function and $A \in R$ be some number. For a given $\varepsilon > 0$ assume

$$A_{\varepsilon}\,(f) \equiv \{x \in I_a^{\infty} : |f\,(x) - A| \geq \varepsilon\}\,.$$

Definition 1. *We say that f has a μ-stat limit A at infinity if and only if*

$$\lim_{x\to\infty} \frac{|A_{\varepsilon}\,(f) \bigcap I_a^x|}{|I_a^x|} = 0,\ \forall \varepsilon > 0,$$

where $I_a^x = [a, x]\,,\ \forall x \in I_a^{\infty}$ and this limit will be denoted as μ-st $\lim\limits_{x\to\infty} f\,(x) = A$.

Let $(I; B; \mu)$ be a measurable space, $I = [a, +\infty)$, where $\mu : B \to R_+$ is a σ-finite measure on $I : \mu\,(I) = +\infty$, B σ-algebra of Borel subsets in I. By $L_p\,(\mu)$, $0 < p < +\infty$, we denote as usual space of measurable (in the sense of $(B; \mu)$) functions $f : I \to R$ with

$$\|f\|_p < +\infty,$$

where

$$\|f\|_p = \begin{cases} \int_I |f\,(t)|^p\, d\mu\,(t)\ \ , & 0 < p < 1, \\ \left(\int_I |f\,(t)|^p\, d\mu\,(t)\right)^{\frac{1}{p}}\ , & 1 \leq p < +\infty. \end{cases}$$

Let $I_x = [a, x]\,,\ \forall x \geq a$. Introduce the following definition.

Definition 2. *Let $|f|^p$, $0 < p < +\infty$, be a locally integrable function on $[a, +\infty)$. We will say that the function f has a $\mu\,[p]$-strong limit (or is $\mu\,[p]$-strong Cesaro summable) at infinity, equal to the number A, if*

$$\lim_{x\to\infty} \frac{1}{\mu\,(I_x)} \int_{I_x} |f\,(t) - A|^p\, d\mu = 0.$$

This limit will be denoted by

$$\mu\,[p] - \lim_{x\to\infty} f\,(x) = A.$$

Note that in the discrete case, p-Cesaro summability has the following form (see, e.g., [1, p. 147])

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} |x_k - \xi|^p = 0,\ 0 < p < +\infty.$$

The theorem below is the μ-analogue of the discrete case.

Theorem 1. *i*) Let the function f be $\mu\,[p]-$ strong Cesaro convergent to A for some $p \in (0, +\infty)$ at infinity. Then $\exists\ \mu - st \lim\limits_{x\to\infty} f\,(x)$ and $\mu - st \lim\limits_{x\to\infty} f\,(x) = A$; *ii*) If $\exists\ \mu - st \lim\limits_{x\to\infty} f = A$ and f is μ-a.e. bounded, then $\exists\ \mu\,[p] - \lim\limits_{x\to\infty} f\,(x)$ and this limit is equal to A.

Note that the obtained results are the generalizations of the results of [2] to the μ-case.

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On the basicity of double system of exponents in the Weighted Lebesgue space

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This work considers the double system of exponentials with linear phase in the weighted space $L_{p,\rho}$ with power weight $\rho(\cdot)$ on the segment $[\pi, \pi]$. Under certain conditions on the weight function $\rho(\cdot)$ and on the perturbation parameters, the completeness, minimality and basicity of this system in $L_{p,\rho}$ is proved.

Consider the following double system of exponents

$$\left\{e^{i[(n+\beta_1)t+\gamma]}; e^{-i[(k+\beta_2)t+\gamma_2]}\right\}_{n\in Z_+;k\in N}, \tag{1}$$

where $\beta_k = Re\beta_k + iIm\beta_k,\ \gamma_k = Re\,\gamma_k + iIm\,\gamma_k,\ k = 1, 2$, are complex parameters, $Z_+ = \{0\} \bigcup N$. We assume that the weight function $\rho(\cdot)$ is of the following power form

$$\rho(t) = \prod_{k=-r}^{r} |t - t_k|^{\alpha_k},$$

where $-\pi = t_{-r} < t_{-r+1} < ... < t_0 = 0 < ... < t_r < \pi$, $\{\alpha_k\}_{k=-\overline{r,r}} \subset R$ are some numbers. We consider the weighted space $L_{p,\rho}$, $1 < p < +\infty$, with the norm $\|\cdot\|_{p,\rho}$:

$$\|f\|_{p,\rho} = \left(\int_{-\pi}^{\pi} |f(t)|^p \rho(t)\,dt\right)^{1/p}.$$

It is easy to see that the basicity properties of the system (1) in $L_{p,\rho}$ are equivalent to the basicity properties of the system

$$\left\{e^{i(n+\beta_1)t}; e^{-i(k+\beta_2)t}\right\}_{n\in Z_+;k\in N}, \tag{2}$$

in $L_{p,\rho}$. We put $g(t) = e^{\frac{i}{2}(\beta_2-\beta_1)t}$. It is evident that $\exists\delta > 0$:

$$0 < \delta \le |g(t)| \le \delta^{-1} < +\infty,\ \forall t \in [-\pi, \pi].$$

Multiplying the system (2) to the function $g(t)$, we immediately obtain from here that the basicity properties of the system (2) on $L_{p,\rho}$ are equivalent to the basicity properties of the following system

$$\left\{e^{i\left(n+\frac{\beta}{2} sign\, n\right)t}\right\}_{n\in Z}, \tag{3}$$

on $L_{p,\rho}$, $\beta=\beta_1+\beta_2$. Thus, the study of basicity properties of the system (1) on $L_{p,\rho}$ is reduced to the investigation of corresponding properties with respect to the system (3) on $L_{p,\rho}$.

We prove the following theorem on the minimality of the system (3).

Theorem 1. *Assume that the following inequalities hold*

$$Re\beta > -1; \quad -1 < \alpha_k < \tfrac{p}{q}, \quad k=\overline{-r+1, r-1};$$

$$-1 < \alpha_{\pm r} < \tfrac{p}{q} + pRe\beta.$$

Then the exponential system $\left\{e^{i\left(n+\frac{\beta}{2} sign\, n\right)t}\right\}_{n\in Z}$ *is minimal in* $L_{p,\rho}$, $1<p<+\infty$.

The following theorem is true.

Theorem 2. *Let* $\rho\in L_1$ *and the parameter* β *satisfy one of the following conditions:*
i) $-Re\beta\in\bigcup_{k=0}^{\infty}(k,k+1)$;
ii) $-\beta\in Z_+$;
iii) $\left|1+e^{it}\right|^{-Re\beta}\in L_{p,\rho}$ *and the following inequalities hold*

$$-1<\alpha_k<\frac{p}{q}, k=-\overline{r,r}.$$

Then the system (3) is complete in $L_{p,\rho}$, *for* $\forall p\geq 1$, *if* $\rho\in L_1$.

By combining the results of Theorems 1 and 2 we arrive at the following conclusion.

Theorem 3. *Let the weight* $\rho(\cdot)$ *and parameter* β *satisfy the following conditions*

$$-1<\alpha_k<\frac{p}{q},\ k=-\overline{r,r};\ \ -1<\alpha_{\pm r}-pRe\beta<\frac{p}{q}.$$

Then the system (3) is complete and minimal in $L_{p,\rho}$, $1<p<+\infty$.

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Polynomials on parabolic manifolds

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Multidimensional parabolic manifolds in terms of a special exhaustion function, perhaps, were first considered in the works of P. Griffiths and J. King (1973), V. Stoll (1977,1980), which were applied in the multidimensional theory of Nevanlinna. Valiron's defective devisors of holomorphic mappings of parabolic manifolds were considered in the work of A. Sadullaev (1980).

Definition 1. *A Stein manifold X is called parabolic if it does not contain a plurisubharmonic function bounded from above, except constants. A Stein manifold X of dimension n is called S parabolic manifold if there exists a special exhaustion function on it $\rho(z)$: $\tilde{A}$) $\rho(z) \in psh(X), \{\rho \leq M\} \subset\subset X \forall M \in \mathbb{R}$; b) ρ is a maximal function outside some compact set K, i.e. $(dd^c\rho)^n = 0$ on $X \backslash K$.*

Further development of the theory of holomorphic and plurisubharmonic functions on parabolic manifolds is connected with the works of many authors (see References). Note that for Riemann surfaces ($dimX = 1$) the concepts of parabolicity and S- parabolicity coincide. But for $dimX > 1$, this question is still open.

On parabolic manifolds, one can define the notion of polynomials.

Definition 2. *If for a function $f(z) \in \mathcal{O}(X)$ there are positive numbers c, d such that for all $z \in X$ the inequality $ln|f(z)| \leq d\rho^+ z + c$ holds, where $\rho^+ z = max\{0, \rho z\}$,then the function f is called ρ-polynomial of degree $\leq d$. The smallest d in this inequality is called the degree of the polynomial f.*

The simplest example of a parabolic manifold is $\mathbb{C}^n$ with exhaust function $\rho(z) = ln|z|$ or any algebraic manifold for which there is a rich class of polynomials. However, we have constructed an example of a parabolic manifold on which there are no non-trivial polynomials except for constants. A manifold X for which the class of polynomials $\mathcal{P}(X)$ is dense in the class of holomorphic functions $\mathcal{O}(X)$ is called *regular*.

In 1962, J. Sichak proved the following generalization of the classical Bernstein-Walsh theorem: let $K \subset \mathbb{C}^n$ is a regular compact and $e_d(f, K) = inf_{p \in \mathcal{P}(\mathbb{C}^n)} sup_{z \in K} |f(z) - p(z)|$ is the minimal deviation of a function f from the class of polynomials $\mathcal{P}(\mathbb{C}^n)$ on K. Then the function f originally defined on a compact K holomorphically extends to a neighborhood $D_R = \{z \in X : \Phi(z, K) < R\}, R > 1$ if and only if $l(f, K) = \overline{lim}_{d \to \infty} e_d^{\frac{1}{d}}(f, K) \leq \frac{1}{R}$. Here $\Phi(z, K) = sup\left\{|f(z)|^{\frac{1}{degf}} : f \in \mathcal{P}(\mathbb{C}^n), \| f \|_K \leq 1\right\}$ is the well known extremal Green's function.

There is the most common

Theorem. *Let $K \subset X$ is a regular compact and X is a regularly parabolic manifold. Then a function $f(z)$ defined on K holomorphically extends to domain $D_R = \{z \in X : \Phi(z, K), R\}, R > 1$ if and only if $l(f, K) \leq \frac{1}{R}$.*

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The scattering problem for the system of six ordinary differential equations on a semi-axis with three given incident waves

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We consider the following system of six first-order ordinary differential equations on the semi-axis $x \geq 0$:

$$-i\frac{dy_k(x)}{dx} + \sum_{j=1}^{6} c_{kj}(x)y_j(x) = \lambda\xi_k y_k(x), \quad k = \overline{1,6} \tag{1}$$

where $||c_{kj}(x)||_{k,j=1}^{6}$ is a matrix with zero diagonal elements, $c_{kk}(x) = 0$, $k = \overline{1,6}$; its elements are measurable complex-valued functions satisfying conditions:

$$|c_{kj}(x)| \leq c \cdot e^{-\varepsilon x}, \quad c > 0, \quad \varepsilon > 0 \tag{2}$$

λ is a spectral parameter; $\xi_1 > \xi_2 > \xi_3 > 0 > \xi_4 > \xi_5 > \xi_6$, $i^2 = -1$.

Any bounded solutions $\{y_1(x,\lambda), \ldots, y_6(x,\lambda)\}$ of the system (1) with coefficients $c_{kj}(x)$ satisfying the conditions (2) assumes the following asymptotic representations on a semi-axis:

$$y_k(x,\lambda) = A_k e^{i\lambda\xi_k x} + o(1), \quad k = 1,2,3, \quad k \to +\infty$$

$$y_k(x,\lambda) = B_k e^{i\lambda\xi_k x} + o(1), \quad k = 4,5,6, \quad k \to +\infty$$

We jointly consider three problems. The k-th problem is to find the solution of the system (1) satisfying the boundary conditions:

$$z_2^k(0,\lambda) = H_k z_1^k(0,\lambda), \quad k = 1,2,3 \tag{3}$$

where

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, H_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{gathered} z_1^k(x,\lambda) = \{y_1^k(x,\lambda), y_2^k(x,\lambda), y_3^k(x,\lambda)\}^t, \\ z_2^k(x,\lambda) = \{y_4^k(x,\lambda), y_5^k(x,\lambda), y_6^k(x,\lambda)\}^t, \quad k = 1,2,3 \end{gathered} \tag{4}$$

(t is transposition) by given asymptotics

$$y_j^k(x,\lambda) = A_j e^{i\lambda\xi_j x} + o(1), \quad k = 1,2,3. \tag{5}$$

Theorem 1. *Let the coefficients* $c_{kj}(x)$ $k,j=\overline{1,6}$ *of the equations system (1) are measurable functions satisfying the conditions (2),* λ *is a real number. Then there is a unique bounded solution of the scattering problem for the system (1) on the semi-axis with given incident waves* $A_1e^{i\lambda\xi_1 x}, A_2e^{i\lambda\xi_2 x}, A_3e^{i\lambda\xi_3 x}$. Moreover,

$$y_j^k(x,\lambda)=B_j^k e^{i\lambda\xi_j x}+o(1), \quad x\to+\infty, \quad k=1,2,3, \quad j=4,5,6. \tag{6}$$

Thus, we can define scattering matrix $S(\lambda)=\{S^1(\lambda),S^2(\lambda),S^3(\lambda)\}$, where

$$S^k(\lambda)=\begin{pmatrix} S_{11}^k(\lambda) & S_{12}^k(\lambda) & S_{13}^k(\lambda) \\ S_{21}^k(\lambda) & S_{22}^k(\lambda) & S_{23}^k(\lambda) \\ S_{31}^k(\lambda) & S_{32}^k(\lambda) & S_{33}^k(\lambda) \end{pmatrix}, k=1,2,3 \tag{7}$$

The scattering problem for system (1) on the whole axis was studied in [1].

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Analysis of blood flow through multiple stenoses in a narrow artery

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A study of the effects of blood flow parameters in narrow arteries having multiple stenoses is discussed here. In this work, blood is considered as a non-Newtonian Kuang-Luo (K-L) fluid model, with no-slip conditions at the arterial wall. In fact, the main properties of the K-L fluid model are that the plasma viscosity and yield stress play a very important role. These parameters make this fluid remarkably similar to blood, however, when we change these parameters the flow characteristics change significantly. We have derived a numerical expression for the blood flow characteristics such as resistance to blood flow, blood flow rate, axial velocity, and skin friction. These numerical expressions have been solved by MATLAB 2021 software and discussed graphically. Furthermore, these results have been compared with Newtonian fluid, and observation made that resistance to blood flow and skin friction is decreased when blood is changed from non-Newtonian to Newtonian fluid.

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$\sigma-$ convergence of order β

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By a φ-function we understood a continuous non-decreasing function $\varphi(u)$ defined for $u \geq 0$ and such that $\varphi(0) = 0, \varphi(u) > 0$, for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$, (see, [9]).

Let φ and f be given φ-function and modulus function, respectively. Moreover, let $T = (t_{nk})(n, k = 1, 2, ...)$ be a real matrix, a lacunary sequence $\theta = (k_r)$ and $0 < \beta \leq 1$ be given. Then we define the following:

$$N_\theta^\beta(T, \varphi, f, \sigma)_0 = \left\{ x = (x_k) : \lim_r \frac{1}{h_r^\beta} \sum_{n \in I_r} f\left(\left| \sum_{k=1}^{\infty} t_{nk} \varphi(|x_{\sigma^k(i)}|) \right| \right) = 0, \text{uniformly in } i \right\},$$

where h_r^β denote the βth power $(h_r)^\beta$ of h_r, that is $h^\beta = (h_r^\beta) = (h_1^\beta, h_2^\beta, h_3^\beta, ...)$.

If $x \in N_\theta^\beta(T, \varphi, f, \sigma)_0$, the sequence x is said to be lacunary strong (T, φ, f, σ)- convergent of order β to zero with respect to a modulus f. Further, we prove some inclusion theorems.

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Some extremal problems of approximation theory of holomorphic functions

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The talk is devoted to some extremal problems of the approximation theory of holomorphic functions in the unit disk. In particular, we discuss Jackson–Stechkin and Bohr–Favard type inequalities.

Let $H^q, 1 \le q \le \infty$, be the Hardy space of functions f holomorphic in the unit disk $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ with standard norm $\|f\|_q < +\infty$. Let $\psi(z) = \sum_{k=1}^{\infty} \psi_k z^k$ be a function holomorphic in $\mathbf{D}$ such that $|\psi_k| > 0$ for all $k \in \mathbf{N}$ and let $H^{\psi,q}$ be the functional class consisting of all functions f holomorphic in $\mathbf{D}$ representable in the form $f = f(0)+g*\psi$, where $g \in H^q$ and $*$ means the Hadamard convolution. We let by f^ψ denote the function g in the previous representation.

If $H^{\psi,q} \subset H^q$ and $n \in \mathbf{N}$ then we can define

$$\begin{aligned} A(n,\psi,q) :=&\quad \sup\left\{\tfrac{E_n(f)_q}{|\psi_n|\|f^\psi\|_q} : f \in H^{\psi,q}, f \not\equiv \text{const}\right\},\\ B(n,\psi,q) :=&\quad \sup\left\{\tfrac{E_n(f)_q}{K^\psi(|\psi_n|,f)_q} : f \in H^{q}, f \not\equiv \text{const}\right\},\\ C(n,\psi,q) :=&\quad \sup\left\{\tfrac{\|f\|_q}{|\psi_n|\|f^\psi f\|_q} : f \in H_n^{\psi,q}, f \not\equiv \text{const}\right\}, \end{aligned}$$

where $K^\psi(|\psi_n|,f)_q := \{\|f-h\|_q + |\psi_n|\|h^\psi\|_q : h \in H^{\psi,q}\}$ is the Petree K–functional and

$$H_n^{\psi,q} := \left\{f \in H^{\psi,q} : f^{(k)}(0) = 0, k = 0,1,\dots,n-1\right\}.$$

To establish the dependence between $A(n,\psi,q)$, $B(n,\psi,q)$ and $C(n,\psi,q)$ we also introduce the following

$$D(n,\psi) := \inf\left\{\|r_n(\psi)+g\|_1 : g \text{ is holomorphic in } \mathbf{D}, g(0) = 0\right\},$$

where

$$r_n(\psi)(z) := 1 + 2\mathrm{Re}\sum_{k=1}^{\infty} \frac{\psi_{n+k}}{\psi_n} z^k.$$

Theorem. *Let $1 \le q \le \infty$ and $n \in \mathbf{N}$. If $D(n,\psi) < +\infty$, then we have the following inequalities*

$$1 \le A(n,\psi,q) = B(n,\psi,q)C(n,\psi,q) \le D(n,\psi).$$

The equality $D(n,\psi) = 1$ holds if and only if $r_n(\psi)(z) \ge 0$ for all $z \in \mathbf{D}$.

Dilation Operators in Besov spaces over Local Fields

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A local field is a locally compact, totally disconnected, non-Archimedian norm valued and non-discrete topological field, we refer [2] to basic Fourier analysis on local fields. The local fields are essentially of two types (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$) namely characteristic zero and of positive characteristic. The characteristic zero local fields include the p-adic field $\mathbb{Q}_p$ and the examples of positive characteristic are the Cantor dyadic groups, Vilenkin p-groups and p-series fields.

Operator theory on local fields, is quite new and lots of new topics are worth to study. In our work [1], we consider the dilation operators of the form

$$(T_k f)(x) = f(q^k x), \qquad x \in K, \quad k \in \mathbb{N}, \tag{1}$$

in the framework of Besov spaces over local fields ($B^s_{rt}(K)$) and more precisely, we prove the following result:

Theorem 1. Let $0 < r, t \leq \infty,\ s > \sigma_r = \max\big(\frac{1}{r} - 1, 0\big)$. For $k \in \mathbb{N},\ T_k$ is defined by (1). Then

$$\|T_k f \mid B^s_{rt}(K)\| \leq (c_2 + c_1 k^{1/t}) q^{k(s-\frac{1}{r})} \|f \mid B^s_{rt}(K)\|,$$

for some c_1, c_2 which are independent of k and for all $f \in B^s_{rt}(K)$.

Note that, in the case of $B^s_{rt}(K)$ (with $s > \sigma_r$), Besov norm of operators T_k (defined by (1)) depends on k, whereas in the case of $B^s_{pq}(\mathbb{R}^n)$, Besov norm of operators $T_k : f \to f(2^k \cdot)$ is independent of the constant k for $s > \sigma_r$ (see [3]).

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Asymmetric natural vibrations of a fluid containing cylindrical shell stiffened with rods and subjected to the action of compressive force along the axis

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The paper studies asymmetric natural vibrations of a fluid-filled cylindrical shell stiffened with rods subjected to the action of compressive force along the axis. Using the constructive-orthotropic model of the shell, compressive models of fluid and contact conditions, a frequency equation for the system under consideration was structured. The obtained equation is a transcendental equation with respect to the desired frequency parameter. Using the logarithmic derivative of the Bessel function, based on the shell thickness and character of change in the stress-strain of the shell, an analytic expression for the vibration frequency of the system under consideration was obtained and the influence of physical and geometrical parameters characterizing the system on these parameters was studied.

The system under consideration consists of a cylindrical shell stiffened with rods and fluid completely filling its inside. Therefore, for studying the vibrations of such a system, we will use equations of motion of a cylindrical shell stiffened with rods, fluid and additional contact conditions.

Fundamentals of the theory of deformation of shells stiffened with rods were given by V.Z.Vlasov [1] and A.I.Lourie [2].

Under the cylindrical shell stiffened with rods we understand a system consisting of a cylindrical shell and rods rigidly stiffened to it along the coordinate lines.

It is considered that the coordinate axes coincide with principal curvature lines and rigidly contact with the shell along these lines. Using the Ostrogradsky-Hamilton variation principle, we can derive the system of main equations of the mentioned system. It is considered that the stress-strain state of the cylindrical shell is completely determined by the equations of the linear theory of shells, based on the Kirchhoff-Liav conjecture. In the calculation of rods, the equations based on Kirchhoff-Clebsh's theory of straight-axis rods are used.

Propagation of small perturbations in the ideal fluid is expressed by the following equation [4]:

$$\nabla^2\Phi - \frac{1}{a^2}\frac{\partial^2\Phi}{\partial t^2} = 0. \tag{1}$$

Here Φ is the fluid's potential, a is the speed of sound propagation in the fluid. In the case of harmonic vibration, equation (1) goes into the Helmholtz equation:

$$\nabla^2\Phi + \frac{\omega^2}{a^2}\Phi = 0. \tag{2}$$

When fluid is incompressible, as $a^2 \to \infty$ the equation (2) goes into the Laplace equation:

$$\nabla^2\Phi = 0.$$

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The Dirichlet problem for semilinear elliptic equations of the second order with discontinuous coefficients

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Let E_n be n-deminsional Euclidean space of the points $x = (x_1, x_2, ..., x_n)$ and Ω be a bounded domain in E_n with boundary $\partial\Omega$ of the class C^2. We consider in Ω the following Dirichlet problem:

$$\sum_{i,j=1}^{2} a_{ij}(x)u_{x_i x_j} + g(x, u_x) = f(x), \ \ x \in \Omega. \tag{1}$$

$$u|_{\partial\Omega} = 0. \tag{2}$$

It is assumed that the coefficients $a_{ij}, \ i, j = 1, 2, ..., n$, of the operator
$L = \sum\limits_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}$ are bounded measurable functions satisfying the conditions

$$\gamma|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \gamma^{-1}|\xi|^2, \ \ \forall \ x \in \Omega, \ \ \forall \ \xi \in E_n, \ \ \gamma \in (0,1), \tag{3}$$

$$\underset{x\in\Omega}{esssup} \frac{\sum\limits_{i,j=1}^{n} a_{ij}^2(x)}{\left(\sum\limits_{i=1}^{n} a_{ii}^2(x)\right)^2} \le \frac{1}{n-1} - \delta, \tag{4}$$

where $\delta \in \left(0, \frac{1}{n}\right)$ is some number and $g(x, u) : \Omega \times E_1 \to E_1$ is a Caratheodory function, measurable with respect to $x \in \Omega$ and for almost all $x \in \Omega$ continuous with respect to $u \in E_1$ and in addition to this, satisfying the following growth condition:

$$|g(x, u_x)| \le b_0(|u_x|^{\mu}), \tag{5}$$

We denote by $\dot{W}_2^2(\Omega)$ the closure of the class of functions $u \in C^{\infty}(\Omega) \cap C(\overline{\Omega})$, $u|_{\partial\Omega} = 0$ with respect to the norm

$$\|u\|_{W_2^2(\Omega)} = \left(\int_{\Omega} \left(u^2 + \sum_{i=1}^{n} u_{x_i}^2 + \sum_{i=1}^{n} u_{x_x x_j}^2\right) dx\right)^{\frac{1}{2}}.$$

The notation $C(...)$ means that the positive constant C depends on the content of parameters included in parenthesis (note, the writing $C(\Omega)$ reads that this constant depends on the smoothness of Ω). A function $u(x) \in \dot{W}_2^2$ is called a strong solution of problem (1)-(2) if it satisfies equation (1) a.e. in Ω.

Theorem 1. *Let $n > 2$, $1 \leq \mu < \frac{n}{n-2}$ and conditions (3)-(5) be satisfied, $\partial\Omega \in C^2$. Then there exists a sufficiently small positive constant $C = C(n, \gamma, \delta, q, b_0, \Omega)$ such that problem (1)- (2) has at least one solution from $\dot{W}_2^2$ for any $f(x) \in L_2(\Omega)$ satisfying*

$$\|f\| \leq C(mes_n\Omega)^{-\left(\frac{n-(n-2)q}{2n(q-1)}\right)}.$$

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Analytic solution to a generalized lifeguard problem

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We give an analytic solution to a generalization of Feynman's lifeguard problem formulated as a cooperative game in which the first object A (lifeguard) is able to move in two adjacent media G, S with maximal speeds $w > 0$ and $v < w$, respectively, and the second object B (swimmer) is able to move in S in a fixed direction with constant speed $u < v$.

Our solution is based on the concept of isochrone $I_A(T)$ defined as the set of points which can be reached by object A at the moment $T > 0$ but cannot be reached at any earlier moment of time. More precisely, we determine the isochrone's exact shape of lifeguard A in the case where the first medium G (ground) is the closed lower half-plane and the second S (sea) is the open upper half-plane. Namely, we give explicit parametric equations of lifeguard's isochrone $I_A(T)$ using its representation as the envelope of a system of circles provided by Huygens principle and prove that, for any $T > 0$, it is a convex piecewise differentiable curve. This enables us to obtain an analytic formula for the minimal rescue time in the case where B moves in a given direction with the constant speed u, and describe the exact shape of the optimal trajectory of lifeguard A. Furthermore, minimizing the minimal rescue time over the unit circle of all possible directions of B we find the optimal collective strategy of both actors A and B, which yields an analytic solution of the problem considered.

We will also formulate several further generalizations of Feynman's lifeguard problem which admit explicit analytic solution using the representation of isochrones as envelopes.

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Lebesgue type inequalities on the sets of (ψ, β)–differentiable functions

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Let $\psi(k)$ be an arbitrary fixed sequence of real nonnegative numbers and let β be a fixed real number.

Denote by $C^{\psi}_{\beta}L_1$ the set of 2π–periodic functions, which for all $x \in \mathbb{R}$ can be represented as convolutions of the form

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \Psi_\beta(x-t)\varphi(t)dt, \ a_0 \in \mathbb{R}, \ \varphi \in L_1, \ \varphi \perp 1 \tag{1}$$

with the generating kernel Ψ_β of the form

$$\Psi_\beta(t) = \sum_{k=1}^{\infty} \psi(k) \cos\big(kt - \frac{\beta\pi}{2}\big), \ \psi(k) \geq 0, \ \beta \in \mathbb{R},$$

such that

$$\sum_{k=1}^{\infty} \psi(k) < \infty.$$

The function φ in equality (1) is called as $(\psi, \beta))$–derivative of the function f and is denoted by f^{ψ}_{β} $(\varphi(\cdot) = f^{\psi}_{\beta}(\cdot))$ [1].

Let $\mathcal{T}_{2n-1}$ be the space of all trigonometric polynomials of degree at most $n-1$ and let $E_n(f)_{L_1}$ be the best approximation of the function $f \in L_1$ in the metric of space L_1, by the trigonometric polynomials t_{n-1} of degree $n-1$, i.e.,

$$E_n(f)_{L_1} = \inf_{t_{n-1} \in \tau_{2n-1}} \|f - t_{n-1}\|_{L_1}.$$

Our aim is to estimate the norms $\|f(\cdot) - S_{n-1}(f;\cdot)\|_{L_1}$ via $E_n(f)_{L_1}$, where $S_{n-1}(f;\cdot)$ are the partial Fourier sums of degree $n-1$ of a function f, by Lebesgue inequalities.

The following theorem takes place.

Theorem 1. *Let $\sum\limits_{k=1}^{\infty} k\psi(k) < \infty$, $\psi(k) \geq 0$, $k = 1, 2, ...,$ $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for an arbitrary function $f \in C^{\psi}_{\beta}L_1$ the following inequality takes place*

$$\|f(\cdot) - S_{n-1}(f;\cdot)\|_C \leq \frac{1}{\pi}\sum_{k=n}^{\infty} \psi(k) E_n(f^{\psi}_{\beta})_{L_1}. \tag{2}$$

Moreover, for arbitrary function $f \in C^{\psi}_{\beta}L_1$ *one can find a function* $\mathcal{F}(x) = \mathcal{F}(f; n, x)$ *from the set* $C^{\psi}_{\beta}L_1$ *such that* $E_n(\mathcal{F}^{\psi}_{\beta})_{L_1} = E_n(f^{\psi}_{\beta})_{L_1}$ *and the following inequality is true*

$$\|\mathcal{F}(\cdot) - S_{n-1}(\mathcal{F}; \cdot)\|_C = \left(\frac{1}{\pi}\sum_{k=n}^{\infty}\psi(k) + \frac{\xi}{n}\sum_{k=1}^{\infty}k\psi(k+n)\right)E_n(f^{\psi}_{\beta})_{L_1}. \tag{3}$$

In (2) the quantity $\xi = \xi(f; n; \psi; \beta)$ *is such that* $-2 \leq \xi \leq 0$.

It should be noticed that the estimates (2) and (3) are asymptotically best possible as $n \to \infty$ in the case when

$$\lim_{n\to\infty}\frac{\frac{1}{n}\sum\limits_{k=1}^{\infty}k\psi(k+n)}{\sum\limits_{k=n}^{\infty}\psi(k)} = 0.$$

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Influence of effect of a sliding friction on the characteristic of natural frequencies of the longitudinally supported cylindrical envelopes with filler at an axial compression

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This thesis is devoted to the study of free vibrations of cylindrical shells with a filler reinforced by discretely distributed longitudinal systems of ribs under axial compression and taking into account friction between the shell and the filler. The analysis of the influence of environmental parameters on the frequency parameters of the natural oscillations of the system.

The problem is solved in an energetic way. The potential energy of the shell loaded with axial compressive forces has the form [1]:

$$E- = \frac{h}{2(1-\nu^2)} \int_0^{\zeta_1} \int_0^{2\pi} \left\{ \left(\frac{\partial u}{\partial \xi} + \frac{\partial \nu}{\partial \theta} - w \right)^2 + 2(1-\nu) \frac{\partial u}{\partial \xi} \left(\frac{\partial \nu}{\partial \theta} - w \right) - \right.$$

$$\left. - \frac{1}{4} \left(\frac{\partial u}{\partial \theta} + \frac{\partial \nu}{\partial \xi} \right)^2 \Bigg] \right\} d\xi d\theta + \frac{h}{24(1-\nu^2)R^2} \int_0^{\xi_1} \int_0^{2\pi} \left(\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \theta^2} + \frac{\partial \nu}{\partial \theta} \right)^2 -$$

$$\left. -2(1-\nu) \left[\frac{\partial^2 w}{\partial \xi^2} \left(\frac{\partial^2 w}{\partial \theta^2} + \frac{\partial \nu}{\partial \theta} \right) - \frac{1}{4} \left(\frac{\partial^2 w}{\partial \xi \partial \theta} + \frac{\partial \nu}{\partial \xi} \right)^2 \right] \right\} d\xi d\theta +$$

$$+ \frac{E_c}{2R} \sum_{l=1}^{k} \int_0^{\xi_1} \left[F_c \left(\frac{\partial u}{\partial \xi} - \frac{h_c}{R} \frac{\partial^2 w}{\partial \xi^2} \right)^2 + \frac{I_{yc}}{R^2} \left(\frac{\partial^2 w}{\partial \xi^2} \right)^2 + \right.$$

$$\left. + \frac{G_c}{E_c} I_{kp.c} \times \frac{G_c}{E_c} I_{kp.c} \left(\frac{\partial^2 w}{\partial \xi \partial \theta} + \frac{\partial \nu}{\partial \xi} \right)^2 \right]_{\theta=\theta_1} d\xi -$$

$$- \frac{\sigma_x h}{2} \int_0^{\xi_1} \int_0^{2\pi} \left(\frac{\partial w}{\partial \xi} \right)^2 d\xi d\theta - \frac{\sigma_x F_c}{2R} \sum_{i=1}^{k} \int_0^{\xi_1} \left(\frac{\partial w}{\partial \xi} \right)^2 \Bigg|_{\theta=\theta_1} d\xi \qquad (1)$$

Here $\xi_1 = \frac{L}{R}$, $\xi = \frac{x}{R}$, $\theta = \frac{y}{R}$ x, y, z- coordinates, E_c, G_c- moduli of elasticity and shear of the material of the longitudinal ribs, k – the number of longitudinal ribs, σ_x - axial compressive stresses, u, ν, w - components of the shell displacement vector, h and R - shell thickness and radius, respectively, E,ν - Young's modulus and Poisson's ratio of the shell material, F_c, I_{yc}, $I_{kp.c}$ - respectively, the area and moments of inertia of the cross-section of the longitudinal rod relative to the axis OX and OZ, as well as the moment of inertia during torsion.

The kinetic energy of the shell is as follows:

$$K = \frac{h}{2(1-\nu^2)} \int_0^{\xi_1} \int_0^{2\pi} \left[\left(\frac{\partial u}{\partial t_1} \right)^2 + \left(\frac{\partial \nu}{\partial t_1} \right)^2 + \left(\frac{\partial w}{\partial t_1} \right) \right] d\xi d\theta$$

$$+ \frac{\overline{\rho_c} E_c F_c}{2R(1-\nu^2)} \sum_{i=1}^{k_1} \int_0^{\xi_1} \left[\left(\frac{\partial u}{\partial t_1} \right)^2 + \left(\frac{\partial w}{\partial t_1} \right) \right]_{\theta=\theta_1} d\xi \tag{2}$$

Here $\overline{\rho_c} = \frac{\rho_c}{\rho_0}$, where ρ_0, ρ_c - the density of the shell materials and the longitudinal rod, respectively, $\theta_i = \frac{2\pi}{k_1} i$.

The interaction of the filler with the shell is represented as a surface load applied to the shell, which performs work on the displacements of the contact surface when the system is transferred from the deformed state to the initial undeformed state.

$$A_0 = -\int_0^{\xi_1} \int_0^{2\pi} (q_x u + q_\theta \nu + q_z w) d\xi d\theta + \int_0^{\xi_1} \int_0^{2\pi} f q_z (u+v) d\xi d\theta \tag{3}$$

where q_x, q_θ, q_z - pressure from the filler side to the shell, f- coefficient of friction.

The equation of motion of the medium in vector form has the form [2,3]:

Where $a^2 t = (\lambda + 2\mu)\rho$, $a^2 e = \mu\rho$, a_t, a_e- the propagation velocity of longitudinal and transverse waves in the aggregate, respectively; $S = S(S_x, S_\theta, S_z)$- displacement vector; λ, μ - Lame coefficients. Contact conditions are added to the systems of equations of motion of the medium (3). It is assumed that the contact between the shell and the aggregate is rigid, i.e. at $r = R$:

$$u = S_x; \quad \nu = S_\theta; \quad w = S_z \tag{4}$$

$$q_x = -\sigma_{rx},\ q_y = -\sigma_{r\theta}, \quad q_z = -\sigma_{rr},\ w = S_r \tag{5}$$

Components σ_{rx}, $\sigma_{r\theta}$, σ_{rr} - stress tensors are defined as follows [2,3,4]:

$$\begin{aligned} \sigma_{rx} &= \mu_s \left(\tfrac{\partial S_x}{\partial r} + \tfrac{\partial S_r}{\partial x} \right); \quad \sigma_{r\theta} = \mu_s \left[r \tfrac{\partial}{\partial r} \left(\tfrac{S_r}{r} \right) + \tfrac{1}{r} \tfrac{\partial S_r}{\partial \theta} \right], \\ \sigma_{rr} &= \lambda_s \left(\tfrac{\partial S_r}{\partial x} + r \tfrac{\partial}{\partial r} \left(\tfrac{S_r}{r} \right) + \tfrac{1}{r} \tfrac{\partial S_\theta}{\partial \theta} \right) + 2\mu_s \tfrac{\partial S_r}{r} \end{aligned} \tag{6}$$

λ_s, μ_s - Lame coefficients for the medium.

Figure 1 shows that with increasing voltage, the system frequency decreases. In addition, the consideration of friction leads to a decrease in the eigenfrequency of the studied structure.

As noted, the methodology for determining the optimal reinforcement parameters is based on comparing the minimum vibration frequencies of the ribbed and smooth cylindrical shells reinforced with longitudinal systems of ribs filled with medium [5].

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Eigensubspaces of resonancing endomorphisms of algebra convergent power series Σ_n

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In this work, the spectral properties of endomorphism acting on the algebra Σ_n of convergent power series of complex variable $z = (z_1, z_2, ..., z_n)$ are studied. We consider in this algebra eigenvalues, eigenfunctions and eigensubspaces of endomorphism $T : \Sigma_n \to \Sigma_n$, $T(f) = f \circ \varphi$ generated by mapping $\varphi : C_n \to C_n$ which modules of all eigenvalues of its linear part at the origin less than one.

Σ_n is the algebra of convergent power series of variables $z = (z_1, z_2, ..., z_n)$ with Σ_n, $n \geq 1$. Let us assume that the endomorphism $T : \Sigma_n \to \Sigma_n$ is resonant, and the eigennumbers of the mapping φ induced to its linear part φ_1 are nonzero, differently and nonresonance. Let is note that the endomorphism T is called resonant if there is a resonance relationship between the eigennumber of φ_1 [4],[2].

Definition 1. *If the numbers $\alpha_1, \alpha_2, ..., \alpha_n$ are the eigenvalues of φ_1 and can be given in the form $\alpha_s = \alpha^m = \alpha_1^{m_1} \cdot \alpha_2^{m_2} \cdot ... \cdot \alpha_n^{m_n}$ (where $m_i \geq 0$, $\sum\limits_{i=1}^{n} m_i \geq 2$), then α_s $(1 \leq s \leq n)$ is called a resonancing eigenvalue [1].*

In the work [3], the eigenfunctions of the nonresonancing endomorphism acting on the algebra Σ_n were determined and the dimensions of the corresponding eigensubspaces were calculated.

Theorem 1. *If modules of eigennumbers $\alpha_1, \alpha_2, ..., \alpha_n$ of the linear part of mapping φ which generated the endomorphism T are less one and nonzero, nonresonancing, differently, then eigenvalues of T have form $\lambda_k = \alpha_1^{k_1} \cdot \alpha_2^{k_2} \cdot ... \cdot \alpha_n^{k_n}$, and corresponding eigensubspaces up to diffeomorphism are generated by the functions $f_k = z^k$. Consequently, all eigensubspaces are one-dimensional.*

Let is consider the case where there is a resonancing relationship between eigennumbers. Let us assume that the numbers $\alpha_1, \alpha_2, ..., \alpha_n$ are the eigenvalues of the linear part φ_1 of the $\varphi : C_n \to C_n$ mapping, which induced to the endomorphism T with a resonancing relation. In this case, the linear operator φ_1 is in the form of *diog* $\varphi_1 = (\alpha_1, \alpha_2, ..., \alpha_n)$. Let is consider a special case. Let us assume that the resonancing relationship is of the form $\alpha_i = \alpha_1^{t_1} \cdot \alpha_2^{t_2} \cdot ... \cdot \alpha_{i-1}^{t_{i-1}} \cdot \alpha_{i+1}^{t_{i+1}} \cdot ... \cdot \alpha_n^{t_n}$. The endomorphism T is $T : f \to f \circ \varphi_1$ by $f \in \Sigma_n$, $f = \sum\limits_{n} a_k z^k$. Here $k = (k_1, k_2, ..., k_n)$ is a multi-index

$$Tf(z) = \lambda f(z). \tag{1}$$

Let is evaluate the equality.

$$Tf(z) = f(\varphi_1(z)) = \sum_{n} a_k \varphi_1^k(z) =$$

$$= \sum_{k_1,k_2,...,k_n} a_{k_1,k_2,...,k_n} (a_1 z_1)^{k_1} (a_2 z_2)^{k_2} ... (a_n z_n)^{k_n}. \quad (2)$$

Let is use the expression. Considering (2) in (1), then

$$\sum_{k_1,k_2,...,k_n} a_{k_1,k_2,...,k_n} \alpha_1^{k_1} \cdot \alpha_2^{k_2} \cdot ... \cdot \alpha_n^{k_n} z_1^{k_1} z_2^{k_2} ... z_n^{k_n} =$$

$$= \lambda \sum_{k_1,k_2,...,k_n} a_{k_1,k_2,...,k_n} z_1^{k_1} z_2^{k_2} ... z_n^{k_n}. \quad (3)$$

Here, if the coefficients $a_u = a_{u_1 u_2 ... u_n}$ and $a_t = a_{t_1 t_2 ... t_n}$ are different from zero, then the eigenvalue $\lambda = \alpha_i^2$ and the corresponding eigenfunction is of the form $f_{(u,t)} = z^u + z^t = z_1^{u_1} z_2^{u_2} ... z_n^{u_n} + z_1^{t_1} z_2^{t_2} ... z_n^{t_n}$. In this case, the following theorem is obtained.

Theorem 2. *If modules of eigennumbers $\alpha_1, \alpha_2, ..., \alpha_n$ of the linear part of mapping φ which generated the endomorphism $T : \Sigma_n \to \Sigma_n$, $T(f) = f \circ \varphi$ are less one and nonzero, resonancing, differently, then eigenvalues of T have form $\lambda = \alpha_i^2$, and corresponding eigensubspaces up to diffeomorphism are generated by the functions $f_{(u,t)} = z^u + z^t$. Consequently, all eigensubspaces are two-dimensional.*

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One approach to approcsimate solutions of stochastic differential equations

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In this study it is shoved that Generalisied Eutropy Optimization Methods (GEOM‘s) represented in over investigations can be successful applied for obtaining approcsimate distributions and density functions of solutions of SDE.

It is known that approcsimate solutions of SDE’s in generally are obtained by using equvedistant times of time interval of given SDE. For example method Euler – Maruyam’a, method Milstein’s.

At fixed time approcsimate solution of SDE arises as random variable dependent on initial random number wich is fixed for each solution.

Each outcome from space of elementary outcomes indicates as simple path.

We have shoved that distributions and density functions of approccimate solutions of SDE at fixed time can be obtained by GEOM’s.

Introduction: Stochastic Differential Equation Modeling has many important applications in population Biology problems, physical systems, stochastic finances problems and other scientific regions. Approcsimate Solving SDE takes general place in solving mentioned problems. It is important to obtain distributions and density functions at fixed time of solutions to SDE. In solving such problems Generalisied Entropy Optimization Methods [4] are applied.

In the represented study it is considired following problems.

In the section 1 generale aspects of GEOM‘s are given.

In the section 2 it is given Euler – Moruyama metod and density functions of SDE by using GEOM’s.

Mentioned method can be used also when other approcsimate methods to solving SDE are applied.

Generalized Entropy Optimization Methods. Shannon entropy measure is defined by formula $H = -\sum\limits_{i=1}^{n} p_i ln p_i$,

Kulubacke lebler measure is defined by formula

$$D\left(p;q\right) = -\sum_{i=1}^{n} p_i ln \frac{p_i}{q_i}.$$

Let us consider H_{max} as functional defined on the set K of moment vector functions

$$g_{(x)}^{(r)} = \left\{ g_1{}_{(x)}^{(r)},\quad g_2{}_{(x)}^{(r)}, \dots g_m{}_{(x)}^{(r)} \right\}, \qquad r = 1, 2, \dots l.$$

Coordinates of $g_{(x)}^{(r)}$ are linearly independent moment functions. Morouver fricvenses of statistical data are $\widehat{p_1}$, $\widehat{p_2}, \ldots, \widehat{p_n}$. Then $\sum_{i=1}^{n} \widehat{p_i} g_{i}{}_{(x_i)}^{(r)} = \mu_j^{(r)}$, $j = 1, 2, \ldots, m^{(r)}$ $r = 1, 2, \ldots, l$ and MaxEnt problem can be formulated as following.

Maximasing function $H(x)$ subject to constraints

$$\sum_{i=1}^{n} p_i g_j^{(r)}(x_i) = \mu_j^{(r)}, \quad j = 1, 2, \ldots, m^{(r)} \quad .$$

If $H_{max} = U$ then $U(g^{(r)})$ is functional defined on K and

$$\min_{1 \le r \le l} U(g^{(r)}) \ = U(\overline{g}), \quad \max_{1 \le r \le l} U(g^{(r)}) \ = U\left(\overline{\overline{g}}\right).$$

$U(\overline{g})$ represents MinMaxEnt distribution and $U\left(\overline{\overline{g}}\right)$ represented MaxMaxEnt distribution in the sense that $\overline{g}$ is moment vector founction obtaining MinMaxEut distribution, and $\overline{\overline{g}}$ obtaines MaxMaxEnt distribution. Theas distributions are Generalisied Entropy optimization distributions are Generalisied Entropy optimization distributions.

In similar form defined MinMinEut and MaxMinxEut distributions if instead considir $D(p; q)$ Kuluback – Leibler measure Methods obtained Generalisied entropy distributions are Colled Generalisied Entropy optimization Methods.

Aprocsimate Solution of Stochastic Differential Equation. An *Itȯ* stochastic differential eqnation on the interval $[0, T]$ has the form

$$X(t, \omega) = X(0, \omega) + \int_0^t f(s, x(s, \omega))\, ds + \int_0^t g(s, x(s, \omega)) dw(s, \omega) \tag{1}$$

for $0 \le t \le T$ where $x(0, \cdot) \in H_{RV}$ or in differential form

$$dX(t, \omega) = f(t, x(t, \omega))\, dt + g(t, x(t, \omega))\, dw(t, \omega) \tag{2}$$

for $0 \le t \le T$ with $x(0, \cdot) \in H_{RV}$.

When apply to (2), Euler‘s method has the form

$$X_{i+1}(\omega) = x_i(\omega) + f(t, x_i(\omega))\, \Delta t + g(t, x_i(\omega))\, \Delta W_i(\omega),$$
$$x_0(\omega) = x(0, \omega). \tag{3}$$

For $i = 0, 1, \ldots, N-1$, where $X_i(0) \approx X(t, \omega)$, $t_i = i\Delta t$, $\Delta t = T/N$, $\Delta W_i(\omega) = (W(t_{i+1}, \omega) - W(t_i, \omega)) \approx N(0,\ \Delta t)$, and where ω indicates a simple path.

If apply Computer Generation Methods [1], [2], [3] of random numbers it is possible generate large numbers from random variable represented by $X_{i+1}(\omega)$ defined in (3). Distributions and density function of mentioned set of generated density function of mentioned set of generated numbers can be obtained by using GEOM‘s [4].

The advantage of GEOM‘s application consists in theirs thinness and more wide applicability.

Conclusion. At fixed equidistant points approcsimate solution is random variable dependent on initial random variable, if apply Computer Generation Methods of random numbers.

It is passible generate large numbers from random variable represented by sample path. Distributions and density function of mentioned set of generated numbers can be fined by using GEOM's. The advantage of GEOM's application consists in theirs thinness and more applicability.

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On flexure of orthotropic rectangular plates with respect to different modularity

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We consider a problem of flexure of orthotropic plates made of different modular materials, simply supported by a rigid contour, loaded with uniformly distributed pressure $q = q_0$.

It is assumed that the axes of the orthotropic material coincide with the directions of the axes x and y.

The relation between stresses and strains in the plane xy has the form [1]:

under tension:

$$\varepsilon_x = \frac{\sigma_x}{E_1^+} - \mu_{21}^+ \frac{\sigma_y}{E_2^+}, \quad \varepsilon_y = \frac{\sigma_y}{E_2^+} - \mu_{12}^+ \frac{\sigma_x}{E_1^+}, \quad \gamma_{xy} = \frac{\tau_{xy}}{G^+}, \tag{1}$$

under compression

$$\varepsilon_x = \frac{\sigma_x}{E_1^-} - \mu_{21}^- \frac{\sigma_y}{E_2^-}, \quad \varepsilon_y = \frac{\sigma_y}{E_2^-} - \mu_{12}^- \frac{\sigma_x}{E_1^-}, \quad \gamma_{xy} = \frac{\tau_{xy}}{G^-}, \tag{2}$$

where E_1^+, E_2^+ and E_1^-, E_2^- are module of elasticity of the material in the direction of the axes x and y, μ_{21}^+, μ_{12}^+ and μ_{21}^-, μ_{12}^- are the Poisson ratios, G^+ and G^- are shear module for tension and compression, respectively.

Note that the following dependencies are satisfied:

$$\mu_{21}^+ E_1^+ = \mu_{12}^+ E_2^+, \quad \mu_{21}^- E_1^- = \mu_{12}^- E_2^-.$$

Solving the equations (1) and (2) with regard to stresses, we obtain:

for tension:

$$\sigma_x = \left(E_1^+\right)^1 \left(\varepsilon_x + \mu_{21}^+ \varepsilon_y\right), \quad \sigma_y = \left(E_2^+\right)^1 \left(\varepsilon_y + \mu_{12}^+ \varepsilon_x\right), \quad \tau_{xy} = G^+ \gamma_{xy}, \tag{3}$$

for compression:

$$\sigma_x = \left(E_1^-\right)^1 \left(\varepsilon_x + \mu_{21}^- \varepsilon_y\right), \quad \sigma_y = \left(E_2^-\right)^1 \left(\varepsilon_y + \mu_{12}^- \varepsilon_x\right), \quad \tau_{xy} = G^- \gamma_{xy}. \tag{4}$$

When deriving a differential equation of flexure of orthotropic plates made of a different modular material, it is accepted that the normal to the median surface only turns, staying straight and we can neglect normal stresses of the plate in the direction of the axis z. In this case, in the layer distant from the median surface by a distance of z, we have dependence between deformations ε_x, ε_y, γ_{xy}, curvatures $\varkappa_x, \varkappa_y$ and torsion λ [2]

$$\varepsilon_x = z\varkappa_x = -z\frac{\partial^2 w}{\partial x^2}, \varepsilon_y = z\varkappa_y = -z\frac{\partial^2 w}{\partial y^2}, \gamma_{xy} = 2z\lambda = -2z\frac{\partial^2 w}{\partial x \partial y} \tag{5}$$

Taking into account (3), (4) and (5), we obtain an expression for bending moments M_x, M_y and the torque M_{xy}:

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz = -\left[\left(D_x^+ + D_x^-\right)\frac{\partial^2 w}{\partial x^2} + \left(D_{21}^+ + D_{21}^-\right)\frac{\partial^2 w}{\partial y^2}\right]$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z dz = -\left[\left(D_y^+ + D_y^-\right)\frac{\partial^2 w}{\partial y^2} + \left(D_{12}^+ + D_{21}^-\right)\frac{\partial^2 w}{\partial x^2}\right] \tag{6}$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} z dz = -2\left(D_{xy}^+ + D_{xy}^-\right)\frac{\partial^2 w}{\partial x \partial y}$$

The equation of equilibrium of rigid plates will be of the form [3]

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q \tag{7}$$

After substituting expressions (6) in equation (7), we obtain the following differential equation of flexure of orthotropic plates with respect to different modularity

$$\left(D_x^+ + D_x^-\right)\frac{\partial^4 w}{\partial x^4} + 2\left[\left(D_1^+ + D_1^-\right) + \left(D_{xy}^+ + D_{xy}^-\right)\right]\frac{\partial^4 w}{\partial x^2 \partial y^2} + \left(D_y^+ + D_y^-\right)\frac{\partial^4 w}{\partial y^4} = q \tag{8}$$

We look for the solution of equation (8) in the form [1]

$$W = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

We expand the lateral load q in double trigonometric series:

$$q = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} q_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

where

$$q_{mn} = \frac{4q_0}{ab}\int_0^a \int_0^b \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} dxdy = \frac{16 q_0}{\pi^2 mn}$$

Behaving as in solving the problem of flexure of orthotropic plates, we obtain a general expression for A_{mn}:

$$A_{mn} = \frac{16 q_0}{\pi^6 mn}\left\{\left(D_x^+ + D_x^-\right)\left(\frac{m}{a}\right)^4 + \right.$$

$$\left. +2\left[\left(D_1^+ + D_1^-\right) + 2\left(D_{xy}^+ + D_{xy}^-\right)\left(\frac{mn}{ab}\right)^2 + \left(D_y^+ + D_y^-\right)\left(\frac{n}{b}\right)^4\right]\right\}^{-1}.$$

As is known, maximum values of the deflection and bending stresses correspond to the center of the plate. In the case when $m = n = 1$, we find:

$$w_{\max} = \frac{16 q_0 a^4}{\pi^6} \left\{ \left(D_x^+ + D_x^-\right) + 2\left[\left(D_1^+ + D_1^-\right) + 2\left(D_{xy}^+ + D_{xy}^-\right)\left(\frac{a}{b}\right)^2 + \left(D_y^+ + D_y^-\right)\left(\frac{a}{b}\right)^4\right]\right\}^{-1}$$

for bending stresses

$$\sigma_{xu}^{\max} = \frac{96 q_0 a^2 \left[\left(D_x^+ + D_x^-\right) + \left(D_1^+ + D_1^-\right)\left(\frac{a}{b}\right)^2\right]}{\pi^4 h^2 \left\{\left(D_x^+ + D_x^-\right) + 2\left[\left(D_1^+ + D_1^-\right) + 2\left(D_{xy}^+ + D_{xy}^-\right)\right]\left(\frac{a}{b}\right)^2 + \left(D_x^+ + D_x^-\right)\left(\frac{a}{b}\right)^4\right\}}$$

$$\sigma_{yu}^{\max} = \frac{96 q_0 a^2 \left[\left(D_y^+ + D_y^-\right)\left(\frac{a}{b}\right)^2 + \left(D_1^+ + D_1^-\right)\right]}{\pi^4 h^2 \left\{\left(D_x^+ + D_x^-\right) + 2\left[\left(D_1^+ + D_1^-\right) + 2\left(D_{xy}^+ + D_{xy}^-\right)\right]\left(\frac{a}{b}\right)^2 + \left(D_y^+ + D_y^-\right)\left(\frac{a}{b}\right)^4\right\}}.$$

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The theory of additions of the rotational motion of a material point around parallel axes in opposite directions by ancient astronomers

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Eudoxus of Cnidus (c. 408 BC - c. 355 BC) can be considered the creator of ancient theoretical astronomy as an independent science.

To explain the irregularities in the movements of the planets, Eudoxus explains their movement with the help of four spheres. In a word, as a result of a combination of the rotational motions of many spheres, according to Eudoxus, one can obtain the direct and reverse motions of the planets, i.e., decompose their apparent complex motions into a combination of simple rotations of the spheres.

As soon as the difficulties in coordinating the observations of the planets made over more than millennia within the framework of Ptolemy's theory became obvious, astronomers of the countries of Islam - including scientists from the Maraga Observatory - made attempts to move on to new models of the world. They rejected the epicycles, sought to revive the system of homocentric spheres of Eudoxus. Nasireddin Tusi, on the other hand, succeeded in constructing a model of such a movement in which the condition of its uniformity would be satisfied in accordance with the observational data. This was achieved again by a certain combination of circular movements. The model itself was presented by its author in the "Reminder on Astronomy" (Tazkira fi ilm al-khay'a), written in Arabic at the Maraga Observatory. "This is the same process by which Eudoxus of Cnidus explained the forward and backward motions of the planets."[1]

Nasireddin Tusi also compiled an exposition of the "Almagest" by Claudius Ptolemy and a number of other astronomical treatises: "Muiniya's Treatise on Astronomy", an addition to it, "The Cream of Knowledge of the Astronomy of the Celestial Spheres." In this cycle of treatises, Nasireddin Tusi builds his own scheme of the kinematics of celestial bodies, different from the Ptolemaic one.

N. Tusi's great contribution to the kinematics of mechanics was the so-called Tusi lemma: if two circles with radii r and 2r are given and the small circle rolls without slipping along the large one, touching it from the inside, then an arbitrary point of contact M of the circle of the small circle makes a rectilinear oscillatory motion along the diameter of the great circle. [3].

Proving this lemma, N. Tusi presented the motion of a small circle as the result of adding two circular motions around parallel axes in different directions. From a modern point of view, we are talking about a complex motion of an absolutely rigid body: there is an addition of two rotations around parallel axes in different directions, and the angular velocity of the relative motion modulo is twice the angular velocity of the portable motion and is directed in the opposite direction; the combination of two such rotations forms the so-called Tusi pair. Unlike an ordinary pair of rotations, in a Tusi pair, the angular velocities of rotations are not equal in absolute value, but differ by a factor of two. If both rotations are uniform, then the point M oscillates harmonically along the diameter of the great circle [2].

Assuming a combination of two uniform rotations with angular velocities equal to ω and -2ω, respectively, N. Tusi obtained the same effect that the introduction of an eccentric gives, i.e., the unevenness of the apparent motion of the planets. Thus, rectilinear motion is obtained by adding two circular motions around parallel axes in different directions.

From the addition of two circular motions around parallel axes in different directions, Nasireddin Tusi receives a reciprocating oscillatory motion along a straight line. In the terminology of vector calculus and the theory of mechanisms, it is described by a pair of vectors rotating at a constant speed, and the speed of the second vector is twice that of the first and directed in the opposite direction. The end of the second vector performs a simple harmonic oscillation along the diameter of the great circle so that the length of the resulting vector periodically changes from zero to 2d.

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Direct and inverse theorems of approximation of functions given on hexagonal domains by the Taylor-Abel-Poisson means

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We present the results of our joint work with professor Jürgen Prestin (University of Lübeck, Germany) and professor Viktor Savchuk (IM NASU). The approximative properties of the Taylor-Abel-Poisson linear summation method of the Fourier series are considered for functions of several variables, periodic with respect to the hexagonal lattice. This type of periodicity is defined by the hexagon lattice given by $\mathcal{H}\mathbb{Z}^2$ (see, for example, [1,2]), where

$$\mathcal{H} = \begin{pmatrix} \sqrt{3} & 0 \\ -1 & 2 \end{pmatrix}, \quad \Omega_{\mathcal{H}} = \left\{ (x_1, x_2) : \ -1 \le x_2, \frac{\sqrt{3}}{2}x_1 \pm \frac{1}{2}x_2 < 1 \right\}.$$

Use the homogeneous coordinates $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3$ such that $t_1 + t_2 + t_3 = 0$ and write $\mathbf{t} \in \mathbb{R}^3_{\mathcal{H}}$. If we set $t_1 = -\frac{x_2}{2} + \frac{\sqrt{3}x_1}{2}$, $t_2 = x_2$, $t_3 := -\frac{x_2}{2} - \frac{\sqrt{3}x_1}{2}$, then $\Omega_{\mathcal{H}}$ becomes $\Omega = \{\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3_{\mathcal{H}} : \ -1 \le t_1, t_2, t_3 < 1\}$, which is the intersection of the plane $t_1 + t_2 + t_3 = 0$ with the cube $[-1, 1]^3$.

A function f is called periodic with respect to the hexagonal lattice $\mathcal{H}$ (or $\mathcal{H}$-periodic) if $f(x) = f(x + \mathcal{H}k)$, $k \in \mathbb{Z}^2$. In homogeneous coordinates, a function $f(\mathbf{t})$ is $\mathcal{H}$-periodic if $f(\mathbf{t}) = f(\mathbf{t} + \mathbf{j})$ whenever $\mathbf{j} \equiv \mathbf{0} \pmod 3$.

Let $L_p = L_p(\Omega)$, $1 \le p \le \infty$, be the space of all functions f, given on the hexagonal domain Ω, with the usual norm $\|f\|_p$. The set $\{\phi_{\mathbf{k}}(\mathbf{t}) = \mathrm{e}^{\frac{2\pi \mathrm{i}}{3}\mathbf{k}\cdot\mathbf{t}} : \ \mathbf{k} \in \mathbb{Z}^3_{\mathcal{H}}\}$ is an orthonormal basis of $L_2(\Omega)$ [1].

Set $\mathbb{J}_\nu := \{\mathbf{k} \in \mathbb{Z}^3_{\mathcal{H}} : |\mathbf{k}| := \max_j\{|k_j|\} = \nu\}$, $\nu = 0, 1, \ldots$, and for any $\varrho \in [0, 1)$ and $r \in \mathbb{N}$, consider the transformation

$$A_{\varrho,r}(f)(\mathbf{t}) := \sum_{\nu=0}^{\infty} \lambda_{\nu,r}(\varrho) \sum_{\mathbf{k}\in\mathbb{J}_\nu} \widehat{f}(\mathbf{k})\phi_{\mathbf{k}}(\mathbf{t}), \quad \widehat{f}(\mathbf{k}) := \langle f, \phi_{\mathbf{k}} \rangle,$$

where for $\nu = 0, 1, \ldots, r-1$, the coefficients $\lambda_{\nu,r}(\varrho) \equiv 1$ and $\lambda_{\nu,r}(\varrho) := \sum_{j=0}^{r-1} \binom{\nu}{j}(1-\varrho)^j \varrho^{\nu-j}$, $\nu = r, r+1, \ldots$

If for a function $f \in L_1(\Omega)$ and $n \in \mathbb{N}$, there exists a function $g \in L_1(\Omega)$ such that $\widehat{g}(\mathbf{k}) = 0$ when $|\mathbf{k}| < n$ and $\widehat{g}(\mathbf{k}) = \frac{|\mathbf{k}|!}{(|\mathbf{k}|-n)!}\widehat{f}(\mathbf{k})$ when $|\mathbf{k}| \ge n$, $\mathbf{k} \in \mathbb{Z}^3_{\mathcal{H}}$, then for the function f, there exists the radial derivative $g =: f^{[n]}$ of order n.

In the space $L_p(\Omega)$, the K–functional of a function f generated by the radial derivative of order $n \in \mathbb{N}$ is the following quantity:

$$K_n(\delta, f)_p := \inf \left\{ \|f - h\|_p + \delta^n \left\| h^{[n]} \right\|_p : h^{[n]} \in L_p(\Omega) \right\}, \quad \delta > 0.$$

Let $\mathcal{Z}_n$, $n \in \mathbb{N}$, denote the set of all continuous strictly increasing functions $\omega(t)$, $t \in [0,1]$, with $\omega(0)=0$ satisfying the following conditions:

$$\int_0^\delta \frac{\omega(t)}{t}\mathrm{d}t = \mathcal{O}(\omega(\delta)), \quad \delta \to 0+, \quad \text{and} \quad \int_\delta^1 \frac{\omega(t)}{t^{n+1}}\mathrm{d}t = \mathcal{O}\left(\frac{\omega(\delta)}{\delta^n}\right), \quad \delta \to 0+.$$

Theorem 1. *Assume that $f \in L_p(\Omega)$, $1 \le p \le \infty$, $n, r \in \mathbb{N}$, $n \le r$ and $\omega \in \mathcal{Z}_n$. Then*

$$\|f - A_{\varrho,r}(f)\|_p = \mathcal{O}\left((1-\varrho)^{r-n}\omega(1-\varrho)\right), \quad \varrho \to 1-,$$

iff there exists the derivative $f^{[r-n]} \in L_p(\Omega)$ and $K_n\left(\delta, f^{[r-n]}\right)_p = \mathcal{O}(\omega(\delta))$, $\delta \to 0+$.

For 2π-periodic functions and functions of several variables 2π-periodic in each variable, similar direct and inverse theorems of approximation by the Taylor-Abel-Poisson means in the integral metrics were given in [3] and [4].

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On basicity of the perturbed system of exponents in weighted Morrey type space

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In this work, a double system of exponents with complex-valued coefficients is considered. Under some conditions on the coefficients and the weight function, it is proved that if this system forms a basis for a weighted Morrey-Lebesgue type space on the interval $[-\pi, \pi]$, it is isomorphic to the classical system of exponents in this space.

Consider the double system of exponents

$$\left\{A(t)\, e^{i\,nt}; B(t)\, e^{-i\,nt}\right\}_{n\in Z_+, k\in N}, \tag{1}$$

with complex valued coefficients $A(t) = |A(t)|\, e^{i\alpha(t)}$; $B(t) = |B(t)|\, e^{i\beta(t)}$ on the interval $[-\pi, \pi]$, where $N-$ is a set of natural numbers, $Z_+ = \{0\} \cup N$.

Let Γ be some rectifiable Jordan curve in the complex plane C. We denote by $|M|_\Gamma$ the linear Lebesgue measure of the set $M \subset \Gamma$.

By the Morrey-Lebesgue space $L^{p,\,\alpha}(\Gamma)$, $0 \le \alpha \le 1$, $p \ge 1$, we mean the normed space of all functions $f(\cdot)$ that are measurable on Γ with a finite norm $\|\cdot\|_{L^{p,\,\alpha}(\Gamma)}$:

$$\|f\|_{L^{p,\,\alpha}(\Gamma)} = \sup_B \left(|B\cap\Gamma|_\Gamma^{\alpha-1} \int_{B\cap\Gamma} |f(\xi)|^p\, |d\xi|\right)^{\frac{1}{p}} < +\infty,$$

where B is an arbitrary ball with center on Γ.

The weight case $L^{p,\,\alpha}_\rho(\Gamma)$ of the Morrey-Lebesgue space with weight function $\rho(\cdot)$ on Γ with the norm $\|\cdot\|_{L^{p,\,\alpha}(\Gamma)}$ is determined in a natural way

$$\|f\|_{L^{p,\,\alpha}_\rho(\Gamma)} = \|f\rho\|_{L^{p,\,\alpha}(\Gamma)}, f \in L^{p,\,\alpha}_\rho(\Gamma).$$

As Γ we will take the unit circle $\gamma = \partial\omega$. Consider the weighted space $L^{p,\,\alpha}_\rho(\gamma) =: L^{p,\,\alpha}_\rho$ with weight $\rho(\cdot)$. In a completely analogous way to the weightless case, we define the space $M^{p,\,\alpha}_\rho$ with weight ρ. Thus, we denote by $M^{p,\,\alpha}_\rho$ the set of functions whose shifts are continuous in $L^{p,\,\alpha}_\rho$, that is

$$\|S_\delta f - f\|_{p,\,\alpha;\,\rho} = \|f(\cdot+\delta) - f(\cdot)\|_{p,\,\alpha;\,\rho} \to 0, \delta \to 0,$$

where S_δ is the shift operator: $(S_\delta f)(x) = f(x+\delta)$.

We denote by $A_{p,\alpha}$ the class of weights $\rho(\cdot)$, for which the singular operator S and the maximal operator M behaves boundedly in the space $L^{p,\alpha}_\rho$.

The following theorem is true.

Theorem 1. *Let $A^{\pm 1}$;$B^{\pm 1} \in L_\infty(-\pi, \pi)$, and $\rho \in A_{p,\alpha}$. If the system (1) forms a basis in $M_\rho^{p,\alpha}$, then it is isomorphic to the classical system of exponents $\left\{e^{int}\right\}_{n\in Z}$ in $M_\rho^{p,\alpha}$, and the isomorphism is given by the operator T_0:*

$$(T_0 f)(t) = A(t) \sum_{n=0}^{\infty} \left(f;\, e^{inx}\right) e^{int} + B(t) \sum_{n=1}^{\infty} \left(f;\, e^{-inx}\right) e^{-int},$$

where

$$(f;\, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, \overline{g(t)} dt.$$

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Solvability of a nonlocal boundary value problem for a multidimensional parabolic equation

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In the mathematical modeling of numerous practice processes, boundary value problems with nonlocal boundary conditions for partial differential equations arise. Nonlocal boundary conditions represent relations connecting the values of the desired solution at the boundary and interior points of the domain. Integral conditions hold a special place among nonlocal boundary conditions. Nonlocal boundary value problems for parabolic equations with integral conditions were studied in [1, 2, 4, 5] and others. Note that such boundary value problems in the classes of generalized solutions are studied least of all.

Suppose that, Ω is a bounded domain in $R^n (n \geq 2)$ with a smooth boundary $S = S' \bigcup S''$, $Q_T = \Omega \times (0,T)$ is a cylinder, $T > 0$ is a given number,
$S_T = S \times (0,T)$ is the lateral surface of the cylinder Q_T, $S'_T = S' \times (0,T)$, $S''_T = S'' \times (0,T)$. The designations of functional spaces and their norms used in this work correspond to [3].

Consider the linear parabolic equation in the cylinder Q_T

$$u_t - \sum_{i,j=1}^{n} \left(a_{ij}(x,t)\, u_{x_j}\right)_{x_i} + a(x,t)\, u = f(x,t), \quad (x,t) \in Q_T. \tag{1}$$

For equation (1), we pose the following boundary value problem: it is required to find a solution subject to the initial condition in the domain Q_T

$$u(x,0) = \varphi(x), \quad x \in \Omega, \tag{2}$$

with boundary condition

$$u|_{S'_T} = 0, \tag{3}$$

and the nonlocal condition

$$\left.\frac{\partial u}{\partial N}\right|_{(x,t)\in S''_T} = \int_{\Omega} K(x,y,t)\, u(y,t)\, dy|_{(x,t)\in S''_T}\,, \tag{4}$$

where $\frac{\partial u}{\partial N} = \sum\limits_{i,j=1}^{n} a_{ij}(x,t)\, u_{x_j} \cos(\nu, x_i)$- is the derivative corresponding to the conormal, ν is the outer normal to the boundary S'', $a_{ij}(x,t)$, $i,j = \overline{1,n}$, $a(x,t)$, $f(x,t)$, $\varphi(x)$, $K(x,y,t)$ are the given measurable functions satisfying the conditions

$$a_{ij}(x,t) = a_{ji}(x,t), \quad i,j = \overline{1,n},$$

$$\nu\xi^2 \le \sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\xi_j \le \mu\xi^2, \quad \xi=(\xi_1,\dots,\xi_n), \quad \xi^2=\sum_{i=1}^{n}\xi_i^2,$$

$$|a(x,t)| \le \mu \quad a.e. \quad Q_T, |K(x,y,t)| \le \mu_1$$

$$a.e. \quad S'' \times \Omega \times (0,T),\, \nu,\, \mu,\, \mu_1 = const > 0, \tag{5}$$

$$\varphi \in L_2(\Omega), \quad f \in L_{2,1}(Q_T). \tag{6}$$

We define a generalized solution $u=u(x,t)$ for the problem (1)-(4) from $V_2^{1,0}(Q_T)$ as an element of $V_{2,0}^{1,0}(Q_T)=\left\{u : u \in V_2^{1,0}(Q_T), \quad u|_{S'_T}=0\right\}$ satisfying the integral identity

$$\int_{Q_T}\left(-u\eta_t+\sum_{i,j=1}^{n} a_{ij}(x,t)\,u_{x_j}\eta_{x_i}+a(x,t)u\eta\right)dxdt-$$

$$-\int_{S''_T}\left[\int_{\Omega} K(s,y,t)\,u(y,t)\,dy\right]\eta(s,t)\,dsdt=$$

$$=\int_{\Omega}\varphi(x)\,\eta(x,0)\,dx+\int_{Q_T} f(x,t)\,\eta dxdt$$

for any function

$$\eta=\eta(x,t)\in \hat{W}_{2,0}^{1,1}(Q_T)=\left\{\eta : \eta \in W_2^{1,1}(Q_T),\ \eta|_{S'_T}=0,\ \eta(x,T)=0,\ x\in\Omega\right\}.$$

Theorem 1. *Let the conditions (5), (6) be satisfied. Then problem (1)-(4) is uniquely solvable in the class $V_2^{1,0}(Q_T)$ and satisfies the estimate*

$$|u|_{Q_T} \equiv \|u\|_{V_2^{1,0}(Q_T)} \le M_1\left(\|\varphi\|_{2,\Omega}+2\|f\|_{2,1,Q_T}\right),$$

here constant M_1 does not depend on φ and f.

Let us show that under several stronger assumptions about the data of problem (1)-(4), generalized solutions from $V_{2,0}^{1,0}(Q_T)$ belong to $W_{2,0}^{1,1}(Q_T)=\left\{u : u\in W_2^{1,1}(Q_T),\ u|_{S'_T}=0\right\}$. Let, in addition to conditions (5) and (6), the following conditions be fulfilled:

$$|a_{ijt}(x,t)| \le \mu_2, i,j=\overline{1,n},\ a.e.\ Q_T,\ |K_t(x,y,t)| \le \mu_3\ a.e.\ S''\times\Omega\times(0,T), \tag{7}$$

$$\varphi \in W_{2,0}^{1}(\Omega),\ f\in L_2(Q_T), \tag{8}$$

where $\mu_2,\mu_3>0$ are some constants.

Theorem 2. *Let conditions (5), (6), (7), (8) be satisfied. Then problem (1)-(4) is uniquely solvable in the class $W_{2,0}^{1,1}(Q_T)$ and satisfies the estimate*

$$\max_{0\le t\le T}\|u_x(x,t)\|_{2,\Omega}^2+\|u_t\|_{2,Q_T}^2 \le M_2\left[\left(\|\varphi\|_{2,\Omega}^{(1)}\right)^2+\|f\|_{2,Q_T}^2\right],$$

where the constant $M_2>0$ does not depend on φ and f.

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Solution of the viscoelastic boundary value problem for a rotating disk

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We solve a problem on a rotating circular disk which properties of the material are described by the relations of the theory of linear isotropic viscoelasticity. The disk without a hole is examined. It is assumed that the thickness of the disk is commensurable with its radius (a thick disk). In this case, the stresses along the thickness change.

Problem statement. We'll use a cylindric system of coordinates $(r,\,\theta,\,z)$, which origin will be located at the centre of the mean surface of the disk. We accept the relations of the theory of linear isotropic viscoelasticity where strains are expressed by stresses [1] as determining relations. In conformity to our problem, these equations will be the following

$$\varepsilon_r = \varepsilon + \int_0^t J_1\left(t-\tau\right)\frac{\partial\left(\sigma_r-\sigma\right)}{\partial\tau}d\tau;\quad \varepsilon_\theta = \varepsilon + \int_0^t J_1\left(t-\tau\right)\frac{\partial\left(\sigma_\theta-\sigma\right)}{\partial\tau}d\tau; \tag{1}$$

$$\varepsilon_z = \varepsilon + \int_0^t J_1\left(t-\tau\right)\frac{\partial\left(\sigma_z-\sigma\right)}{\partial\tau}d\tau;\ \varepsilon_{rz} = \int_0^t J_1\left(t-\tau\right)\frac{\partial\sigma_{rz}}{\partial\tau}d\tau, \tag{2}$$

$$\varepsilon = \int_0^t J_2\left(t-\tau\right)\frac{\partial\sigma}{\partial\tau}d\tau. \tag{3}$$

Here $\sigma_r, \sigma_\theta, \sigma_z, \sigma_{rz}$ are the stress components; $\varepsilon_r, \varepsilon_\theta, \varepsilon_z, \varepsilon_{rz}$ are the strains components; $\sigma = \left(\sigma_r+\sigma_\theta+\sigma_z\right)/3$ is a mean stress; $\varepsilon = \left(\varepsilon_r+\varepsilon_\theta+\varepsilon_z\right)/3$ is a mean strain; t is time; the functions $J_1\left(t\right)$ and $J_2\left(t\right)$ are independent, isotropic, creeping functions. The function $J_1\left(t\right)$ corresponds to the shift state, the function $J_2\left(t\right)$ to the volumetric expansion state. The stress and strain components are the functions of cylindrical coordinates r and z. The latters are omitted when writing (1),(2).

The equilibrium equations should be satisfied [2]

$$\frac{\partial\sigma_r}{\partial r} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_r-\sigma_\theta}{r} + \rho\omega^2\left(t\right)r = 0;\quad \frac{\partial\sigma_{rz}}{\partial r} + \frac{\partial\sigma_z}{\partial z} + \frac{\sigma_{rz}}{r} = 0, \tag{4}$$

Where ρ is a mass of volume unit of the disk material, $\omega = \omega\left(t\right)$ is angular speed of rotation. Strain compatibility conditions are also should be satisfied [2]:

$$\frac{\partial^2\varepsilon_\theta}{\partial r^2} + \frac{1}{r}\left(2\frac{\partial\varepsilon_\theta}{\partial r} - \frac{\partial\varepsilon_r}{\partial r}\right) = 0;\quad \frac{\partial^2\varepsilon_\theta}{\partial z^2} + \frac{1}{r}\left(\frac{\partial\varepsilon_z}{\partial r} - 2\frac{\partial\varepsilon_{rz}}{\partial z}\right) = 0; \tag{5}$$

$$-2\frac{\partial^2\varepsilon_{rz}}{\partial r\partial z} + \frac{\partial^2\varepsilon_z}{\partial r^2} + \frac{\partial^2\varepsilon_r}{\partial z^2} = 0;\quad \frac{\partial^2\varepsilon_\theta}{\partial r\partial z} + \frac{1}{r}\left(\frac{\partial\varepsilon_\theta}{\partial z} - \frac{\partial\varepsilon_r}{\partial z}\right) = 0\,. \tag{6}$$

The strain tensor and displacement vector components are connected with Cauchy's kinematic equations [2]

$$\varepsilon_r = \frac{\partial u_r}{\partial r}; \quad \varepsilon_\theta = \frac{u_r}{r}; \quad \varepsilon_z = \frac{\partial u_z}{\partial z}; \quad \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \tag{7}$$

Where u_r and u_z are the displacement vector components.

According to [2] we accept the boundary condition in the form

$$\int_{-l}^{+l} \sigma_r dz = 0, \text{ for } r = a. \tag{8}$$

Where $2l$ is the thickness, a is the disk's radius, we accept that the resultant of radial stresses on the lateral surface equals zero. Here, as in the case of the elastic problem, the solution will determine the state of the whole disk except the parts close to the lateral surface.

Problem solution. The solution of the problem was obtained using the method presented in [3]. The following theorem is proved.

Theorem. *The viscoelastic problem (1)-(8) has only one solution and the form is as follows:*

$$\sigma_r = \frac{3}{8}\left(a^2 - r^2\right)\rho\omega^2(t) + \frac{1}{8}\left(a^2 - r^2\right)A(t) + \frac{1}{6}\left(l^2 - 3z^2\right)B(t);$$

$$\sigma_\theta = \frac{1}{8}\left(3a^2 - r^2\right)\rho\omega^2(t) + \frac{1}{8}\left(a^2 - 3r^2\right)A(t) + \frac{1}{6}\left(l^2 - 3z^2\right)B(t);$$

$$\sigma_z = 0; \quad \sigma_{rz} = 0.$$

$$u_r = \frac{r}{8}\left[a^2\left(3\varphi_1(t) - 2\varphi_2(t) - \varphi_3(t)\right) - r^2\left(\varphi_1(t) - \varphi_3(t)\right)\right] +$$

$$+\frac{r}{6}\left(l^2 - 3z^2\right)\left(\varphi_2(t) + \varphi_3(t)\right);$$

$$u_z = -\frac{z}{4}\left[a^2\left(3\varphi_2(t) + \varphi_3(t)\right) - 2r^2\left(\varphi_2(t) + \varphi_3(t)\right)\right] - \frac{z}{3}\left(l^2 - z^2\right)\left(\varphi_4(t)\right. .$$

The functions $A(t)$, $B(t)$, $\varphi_i(t)$ $\left(i = \overline{1,4}\right)$ contained in these formulas are determined through the given function $\omega(t)$ by the relations

$$A(t) = \frac{1}{2}\omega_1(t) - \frac{3}{2\left(2J_1(0) + J_2(0)\right)}\left[\int_0^t J_2(t-\tau)\, d\omega_1(\tau) - \right.$$

$$\left. - \int_0^t \eta_R(t-\tau)\int_0^t J_2(\tau - \xi)\, d\omega_1(\xi)\, d\tau\right],$$

$$B(t) = 2\omega_1(t) - A(t) - \frac{6}{J_1(0) + 2J_2(0)}\left[\int_0^t J_2(t-\tau)\, d\omega_1(\tau) - \right.$$

$$\left. - \int_0^t \Omega_R(t-\tau)\int_0^\tau J_2(\tau - \xi)\, d\omega_1(\xi)\, d\tau\right].$$

$$\varphi_1(t) = \frac{1}{3}\int_0^t [2J_1(t-\tau) + J_2(t-\tau)]\, d\omega_1(\tau)\,;\ \varphi_2(t) = \frac{1}{3}\int_0^t [J_1(t-\tau) - J_2(t-\tau)]\, d\omega_1(\tau)\,;$$

$$\varphi_3(t) = \frac{1}{3}\int_0^t [J_1(t-\tau) - J_2(t-\tau)]\, dA(\tau)\quad \varphi_4(t) = \frac{1}{3}\int_0^t (J_1(t-\tau) - J_2(t-\tau))\, dB(\tau)$$

Here $\omega_1(t) = \rho\omega^2(t)$; $\eta_R(t)$ and $\Omega_R(t)$ are the resolvents of the functions $\eta(t)$ and $\Omega(t)$:

$$\eta(t) = \frac{1}{2J_1(0) + J_2(0)}\frac{d}{dt}(2J_1(t) + J_2(t))\,,\ \Omega(t) = \frac{1}{J_1(0) + 2J_2(0)}\frac{d}{dt}(J_1(t) + 2J_2(t))$$

The resolvents $\eta_R(t)$ and $\Omega_R(t)$ may be determined by means of series composed of iterated kernels [4]. For instance, for the resolvent $\eta_R(t)$ we have: $\eta_R(t) = \sum\limits_{n=0}^{\infty} \eta_{n+1}(t)$; $\eta_{n+1}(t) = \int_0^t \eta_1(t-\tau)\ \eta_n(\tau)\, d\tau,\ n = 1, 2, ...;\ \eta_1(t) = \eta(t)$. According to [4], the series determining the resolvent $\eta_R(t)$ converges absolutely and uniformly on the given time interval. We can also write formulae for the resolvent $\Omega_R(t)$.

The strain tensor components are determined by the formulas (5). Wherein $\varepsilon_r, \varepsilon_\theta, \varepsilon_z$ are different from zero and are expressed in terms of functions $A(t)$, $B(t)$, $\varphi_i(t)$ $\left(i = \overline{1,4}\right)$. Besides, $\varepsilon_{rz} = 0$.

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Structure of essential spectra and discrete spectrum of the energy operator of six-electron systems in the Hubbard model. second singlet state

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We consider the energy operator of six-electron systems in the Hubbard model and investigate the structure of essential spectrum and discrete spectrum of the system in the second singlet state. Hamiltonian considering system has the form [1,2]

$$H = A\sum_{m,\gamma} a^+_{m,\gamma}a_{m,\gamma} + B\sum_{m,\tau,\gamma} a^+_{m,\gamma}a_{m+\tau,\gamma} + U\sum_{m} a^+_{m,\uparrow}a_{m,\uparrow}a^+_{m,\downarrow}a_{m,\downarrow}. \tag{1}$$

Here A is the electron energy at a lattice site, B is the transfer integral between neighboring sites, $\tau = \pm e_j,\ j = 1, 2, ..., \nu$, where e_j are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors, U is the parameter of the on-site Coulomb interaction of two electrons, γ is the spin index, $\gamma =\uparrow$ or $\gamma =\downarrow$, $\uparrow$ and $\downarrow$ denote the spin values $\frac{1}{2}$ and $-\frac{1}{2}$, and $a^+_{m,\gamma}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^\nu$.

The six electron systems have octet states, quintet states, triplet states, and singlet states. The energy of the system depends on its total spin S. Along with the Hamiltonian, the N_e electron system is characterized by the total spin S, $S = S_{max}, S_{max}-1, ..., S_{min}, S_{max} = \frac{N_e}{2}, S_{min} = 0, \frac{1}{2}$.

Hamiltonian (1) commutes with all components of the total spin operator $S = (S^+, S^-, S^z)$, and the structure of eigenfunctions and eigenvalues of the system therefore depends on S. The Hamiltonian H acts in the antisymmetric Fock space $\mathtt{H}_{as} = l_2^{as}((Z^\nu)^6)$, where $l_2^{as}((Z^\nu)^6)$ is the subspace of antisymmetric functions of $l_2((Z^\nu)^6)$.

Let φ_0 be the vacuum vector in the space $\mathtt{H}_{as}$. The second singlet state corresponds to the free motion of six electrons over the lattice and their interactions with the basic functions ${}^2s^0_{p,q,r,t,k,n\in Z^\nu} = a^+_{p,\uparrow}a^+_{q,\downarrow}a^+_{r,\uparrow}a^+_{t,\downarrow}a^+_{k,\uparrow}a^+_{n,\downarrow}\varphi_0$. The subspace ${}^2\mathtt{H}^0_s$, corresponding to the second singlet state is the set of all vectors of the form ${}^2\psi^0_s = \sum_{p,q,r,t,k,n\in Z^\nu} f(p,q,r,t,k,n)\,{}^2s^0_{p,q,r,t,k,n\in Z^\nu}, f \in l_2^{as}$, where l_2^{as} is the subspace of antisymmetric functions in the space $l_2((Z^\nu)^6)$. We denote by ${}^2H^0_s$ the restriction of operator H to the subspace ${}^2\mathtt{H}^0_s$.

Theorem 1. *The subspace ${}^2\mathtt{H}^0_s$ is invariant under the operator H, and the restriction ${}^2H^0_s$ of operator H to the subspace ${}^2\mathtt{H}^0_s$ is a bounded self-adjoint operator. It generates a bounded self-adjoint operator ${}^2\overline{H}^0_s$ acting in the space l_2^{as} as*

$${}^2\,\overline{H}^0_s\,{}^2\psi^0_s = 6Af(p,q,t,t,k,n) + B\sum_{\tau}[f(p+\tau,q,r,t,k,n) + f(p,q+\tau,r,t,k,n)+$$

$$+f(p,q,r+\tau,t,k,n)+f(p,q,r,t+\tau,k,n)+f(p,q,r,t,k+\tau,n)+f(p,q,r,k,n+\tau)]+$$
$$+U[\delta_{p,q}+\delta_{q,r}+\delta_{p,t}+\delta_{q,k}+\delta_{p,n}+\delta_{r,n}+\delta_{r,t}+\delta_{k,n}+\delta_{t,k}]f(p,q,r,t,k,n). \quad (2)$$

The operator ${}^2H_s^0$ *acts on a vector* ${}^2\psi_s^0 \in {}^2\mathtt{H}_s^0$ *as*

$$ {}^2H_s^0\,{}^2\psi_s^0 = \sum_{p,q,r,t,k,n\in Z^\nu} ({}^2\overline{H}_s^0 f)(p,q,r,t,k,n)^2 s^0_{p,q,r,t,k,n\in Z^\nu}. \quad (3)$$

Lemma 1. *The spectra of the operators* ${}^2H_s^0$ *and* ${}^2\overline{H}_s^0$ *coincide.*

We call the operator ${}^2H_s^0$ the six-electron second singlet state operator in the Hubbard model.

Let $\mathtt{F} : l_2((Z^\nu)^6) \to L_2((T^\nu)^6) \equiv {}^2\widetilde{\mathtt{H}}_s^0$ be the Fourier transform, where T^ν is the $\nu-$ dimensional torus endowed with the normalized Lebesgue measure $d\lambda$, i.e. $\lambda(T^\nu) = 1$. We set ${}^2\widetilde{H}_s^0 = \mathtt{F}\,{}^2\overline{H}_s^0\mathtt{F}^{-1}$. In the quasimomentum representation, the operator ${}^2\overline{H}_s^0$ acts in the Hilbert space $L_2^{as}((T^\nu)^6)$, where L_2^{as} is the subspace of antisymmetric functions in $L_2((T^\nu)^6)$.

Theorem 2. *The Fourier transform of operator* ${}^2\overline{H}_s^0$ *is an operator* ${}^2\widetilde{H}_s^0 = \mathtt{F}\,{}^2\overline{H}_s^0\mathtt{F}^{-1}$ *acting in the space* ${}^2\widetilde{\mathtt{H}}_s^0$ *be the formula*

$$ {}^2\widetilde{H}_s^0\,{}^2\psi_s^0 = h(\lambda,\mu,\gamma,\theta,\eta,\xi)f(\lambda,\mu,\gamma,\theta,\eta,\xi) + U\int_{T^\nu}[f(t,\lambda+\mu-t,\gamma,\theta,\eta,\xi)+$$
$$+f(t,\mu,\gamma,\lambda+\theta-t,\eta,\xi)+f(t,\mu,\gamma,\theta,\eta,\lambda+\xi-t)+f(\lambda,t,\mu+\gamma-t,\theta,\eta,\xi)+$$
$$+f(\lambda,t,\gamma,\theta,\mu+\eta-t,\xi)+f(\lambda,\mu,t,\gamma+\theta-t,\eta,\xi)+f(\lambda,\mu,t,\theta,\eta,\gamma+\xi-t)+$$
$$+f(\lambda,\mu,\gamma,t,\theta+\eta-t,\xi)+f(\lambda,\mu,\gamma,\theta,t,\eta+\xi-t)]dt, \quad (4)$$

where $h(\lambda,\mu,\gamma,\theta,\eta,\xi) = 6A + 2B\sum_{i=1}^{\nu}[cos\lambda_i + cos\mu_i + cos\gamma_i + cos\theta_i + cos\eta_i + cos\xi_i]$.

Theorem 3. *a). Let* $\nu = 1$, *and* $U < 0$, *then the essential spectrum of operator* ${}^2\widetilde{H}_s^0$ *is the union of seven segments:* $\sigma_{ess}({}^2\widetilde{H}_s^0) = [a+c+e, b+d+f] \cup [a+c+z_3, b+d+z_3] \cup [a+e+z_2, b+f+z_2] \cup [a+z_2+z_3, b+z_2+z_3] \cup [c+e+z_1, d+f+z_1] \cup [c+z_1+z_3, d+z_1+z_3] \cup [e+z_1+z_2, f+z_1+z_2]$, *and discrete spectrum of operator* ${}^2\widetilde{H}_s^0$ *is consists of a unique eigenvalue:* $\sigma_{disc}({}^2\widetilde{H}_s^0) = \{z_1+z_2+z_3\}$, *what lies to the below than the left edge of the essential spectrum of operator* ${}^2\widetilde{H}_s^0$*. Here, and hereafter* $a = 2A - 4B\cos\frac{\Lambda_1}{2}$, $b = 2A + 4B\cos\frac{\Lambda_1}{2}$, $c = 2A - 4B\cos\frac{\Lambda_2}{2}$, $d = 2A + 4B\cos\frac{\Lambda_2}{2}$, $e = 2A - 4B\cos\frac{\Lambda_3}{2}$, $f = 2A + 4B\cos\frac{\Lambda_3}{2}$, $z_1 = 2A - \sqrt{9U^2 + 16B^2cos^2\frac{\Lambda_1}{2}}$, $z_2 = 2A - \sqrt{9U^2 + 16B^2cos^2\frac{\Lambda_2}{2}}$, $z_3 = 2A - \sqrt{9U^2 + 16B^2cos^2\frac{\Lambda_3}{2}}$.

b). Let $\nu = 1$, *and* $U > 0$, *then the essential spectrum of operator* ${}^2\widetilde{H}_s^0$ *is the union of seven segment's:* $\sigma_{ess}({}^2\widetilde{H}_s^0) = [a+c+e, b+d+f] \cup [a+c+\widetilde{z}_3, b+d+\widetilde{z}_3] \cup [a+e+\widetilde{z}_2, b+f+\widetilde{z}_2] \cup [a+\widetilde{z}_2+\widetilde{z}_3, b+\widetilde{z}_2+\widetilde{z}_3] \cup [c+e+\widetilde{z}_1, d+f+\widetilde{z}_1] \cup [c+\widetilde{z}_1+\widetilde{z}_3, d+\widetilde{z}_1+\widetilde{z}_3] \cup [e+\widetilde{z}_1+\widetilde{z}_2, f+\widetilde{z}_1+\widetilde{z}_2]$, *and discrete spectrum of operator* ${}^2\widetilde{H}_s^0$ *is consists of a unique eigenvalue:* $\sigma_{disc}({}^2\widetilde{H}_s^0) = \{\widetilde{z}_1+\widetilde{z}_2+\widetilde{z}_3\}$, *what lies to the above than the right* *edge of the essential spectrum of operator* ${}^2\widetilde{H}_s^0$. *Here* $\widetilde{z}_1 = 2A + \sqrt{9U^2 + 16B^2 cos^2 \frac{\Lambda_1}{2}}$, $\widetilde{z}_2 = 2A + \sqrt{9U^2 + 16B^2 cos^2 \frac{\Lambda_2}{2}}$, $\widetilde{z}_3 = 2A + \sqrt{9U^2 + 16B^2 cos^2 \frac{\Lambda_3}{2}}$.

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Korovkin-type theorems for some types of convergence

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Let (X,d) and (Y,ρ) be metric spaces, (f_n) be a sequence of functions from X to Y and f be a function from X to Y. Let us give the definitions of alpha convergence, semi-alpha convergence and exhaustiveness in the following.

Definition 1.[1] *The sequence (f_n) alpha converges to f, if for every $x \in X$ and for every sequence (x_n) of points of X converging to x, the sequence $(f(x_n))$ converges to $f(x)$.*

Definition 2.[2] *The sequence (f_n) is called exhaustive at $x_0 \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $x \in B_d(x_0,\delta)$ and all $n \geq n_0$ we have that $\rho(f_n(x), f_n(x_0)) < \varepsilon$.*

Definition 3.[3] *Let $x_0 \in X$. The sequence (f_n) semi-alpha converges to f at x_0, it is denoted by $f_n \xrightarrow{semi-\alpha} f$, if*

1. *$f_n(x_0) \to f(x_0)$.*
2. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m \geq n$ and $\rho(f_m(x), f(x_0)) < \varepsilon$ for all $x \in B_d(x_0,\delta)$.*

In this study, we investigate the Korovkin-type theorems depending upon the types of convergence such as alpha convergence, semi-alpha convergence and the notion of exhaustiveness defined above. Since it is known that the convergence types mentioned above are between point-wise convergence and uniform convergence, it will be observed that the conditions can be alleviated in Korovkin's theorem.

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Differential game in occasion of L-catch for non-inertial players under non-stationary geometric constraints on controls

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The present paper is devoted to the L-catch and evasion games with two players, called pursuer and evader, whose controls adhere to non-stationary geometric constraints of various types. Such problems are quite relevant for the processes where the rates of control parameters fluctuate consistently during the time. First, the pursuit problem is discussed and a pursuer strategy guaranteeing the L-catch is defined using the method of Chikrii's resolving functions. Then, the evasion problem is dealt with by means of a specific control function of the evader.

Consider the game including the players P (pursuer) and E (evader). If x, y are their state vectors and u, v are their velocity vectors, respectively in $\mathbf{R}^n$, then the motion dynamics of the players are described by

$$P: \quad \dot{x} = u, \quad x(0) = x_0, \tag{1}$$

$$E: \quad \dot{y} = v, \quad y(0) = y_0, \tag{2}$$

where $x, y, u, v \in \mathbf{R}^n$, $n \geq 1$; x_0 and y_0 are the initial states of P and E which are given under $|x_0 - y_0| > L$, $L > 0$. The vectors u and v act as the control parameters and they are taken as measurable functions $u(\cdot) : [0, +\infty) \to \mathbf{R}^n$ and $v(\cdot) : [0, +\infty) \to \mathbf{R}^n$ being subjected to the *geometric constraints* [3, 5] (or the *G-constraints*)

$$|u(t)| \leq \rho(t) \text{ for almost every } t \geq 0, \tag{3}$$

$$|v(t)| \leq \sigma(t) \text{ for almost every } t \geq 0, \tag{4}$$

where $\rho(t)$ and $\sigma(t)$ are non-negative integrable functions designating the maximal speeds of P and E. Let $\mathbb{U}$ be the class of all measurable functions $u(\cdot)$ corresponding to (3), and let $\mathbb{V}$ be the class of all measurable functions $v(\cdot)$ satisfying (4).

Definition 1. *A measurable function* $u(\cdot) = (u_1(\cdot), ..., u_n(\cdot)) \in \mathbb{U}$ $\Big(v(\cdot) = (v_1(\cdot), ..., v_n(\cdot)) \in \mathbb{V}\Big)$ *is called an admissible control of the player P (of the player E).*

If $u(\cdot) \in \mathbb{U}$ and $v(\cdot) \in \mathbb{V}$, then the solutions to Cauchy's problems (1)-(2) are

$$x(t) = x_0 + \int_0^t u(s)ds, \quad y(t) = y_0 + \int_0^t v(s)ds$$

which are called the *trajectories of the players P and E*, appropriately.

The main target of P is to gain ground on E at the distance $L > 0$ (the L-catch problem), i.e. to achieve

$$|x(\eta) - y(\eta)| \leq L \tag{5}$$

at a finite time $\eta > 0$. Whereas the objective of E is to avoid (5) (the evasion problem), i.e. to maintain the condition $|x(t) - y(t)| > L$ for any $t \geq 0$ or if this can not be done, then E strives to put off a time of the occurrence of (5).

Let us take as $z(t) = x(t) - y(t)$, $z_0 = x_0 - y_0$.

Now we define a strategy for P on the basis of the works [1, 2, 4].

Definition 2. *For $\rho(t) \geq \sigma(t)$, we call the function*

$$\boldsymbol{U}(z_0, t, v) = v + \gamma(z_0, t, v)\,(\xi(z_0, t, v) - z_0) \tag{6}$$

the L-catch strategy or Π_L-strategy for P, where

$$\gamma(z_0, t, v) = \frac{1}{|z_0|^2 - L^2}\cdot$$

$$\cdot\left[\langle v, z_0\rangle + \rho(t)L + \sqrt{(\langle v, z_0\rangle + \rho(t)L)^2 + (|z_0|^2 - L^2)(\rho^2(t) - |v|^2)}\right],$$

$\xi(z_0, t, v) = -(v - \gamma(z_0, t, v)z_0)/|v - \gamma(z_0, t, v)z_0|L$, and $\langle v, z_0\rangle$ is the scalar product of the vectors v and z_0 in $\mathbf{R}^n$.

Definition 3. *It is said that Π_L-strategy (6) guarantees the L-catch on the time interval $\left[0, T(z_0, v(\cdot))\right]$ if for any $v(\cdot) \in \mathbb{V}$:*

a) there exists an instant $t^* \in \left[0, T(z_0, v(\cdot))\right]$ such that $|z(t^*)| \leq L$;

b) $\boldsymbol{U}(z_0, t, v(\cdot)) \in \mathbb{U}$ for all $t \in \left[0, t^*\right]$, where we say the number $T(z_0, v(\cdot))$ a guaranteed time of the L-catch.

Theorem 1. *Let $\rho(t) \geq \sigma(t)$ for all $t \geq 0$. Then Π_L-strategy (6) guarantees the L-catch at a time $T(z_0, v(\cdot)) \leq T_L$ from arbitrary point $z_0 \notin LS$, where S is the unit ball centered at zero point of the space $\mathbf{R}^n$, and T_L is the first positive root of*

$$\int_0^t (\rho(s) - \sigma(s))ds = |z_0| - L.$$

Theorem 2. *If $\int_0^t (\rho(s) - \sigma(s))ds < |z_0| - L$ for all $t \geq 0$, then the control function $\boldsymbol{V}(t) = -\sigma(t)z_0/|z_0|$ guarantees that $|z(t)| > L$ for each $t \geq 0$.*

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The generalized Lengyel-Stone criterion and its applications

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Our goal is to prove in Banach space the criterion of Lengyel -Stone (CLS) on the bounds of the Hermitian operator in Hilbert space. This is historically the first criterion of the topological closeness of the numerical range of an operator. This issue was marked in the list of problems in the book [1, p.129].

Problem no 8. For what operators T and spaces X $V(T)$ is a closed set?

Here $V(T)$ is a spatial or Baur's numerical range. Lengyel-Stone [2, p.84] have proved that for the Hermitian operator T in Hilbert space the upper (lower) bounds of the Hausdorff numerical range $W(T)$ [3, ch.17] are attained if they are eigenvalues for T.

We will prove that CLS is valid for the norm-Hermitian operators in some classes of Banach spaces and is not always valid in any B space.

Theorem 1. *For the norm-Hermitian operator, $T \in B(x)$ (algebra of bounded linear operators) in round Banach space X lower (upper) bound of $V(T)$ is attained if it is the eigenvalue of T. Moreover, the condition of roundedness X can not be omitted.*

Theorem 2. *Theorem 1 is also valid for the norm-Hermitian operators in reflexive smooth space X.*

Theorem 3. *In a reflexive round (or reflexive smooth) X the bounds $V(T)$ are attained if they do not lie in the continuous spectrum of the operator T.*

1). The first application of theorems 1-3 to the norm Hermitian operators.

Theorem 4. *In the reflexive X the norm-Hermitian S for $0 \notin V(S)$ will be quasiinvertible and definite. If X is round, then S is also definite and injective, then $0 \notin V(S)$.*

Theorem 5. *In any X the normly-Hermitian with the condition $0 \notin \overline{V}(S)$ will be invertible and definite. If in uniform round X the norm-Hermitian S is definite and invertible, then $0 \notin \overline{V}(S)$.*

2). The second application of the generalized CLS is transfer on Banach spaces two Meng's criteria on the closeness of Hausdorff numerical range of normal and unitary operators in Hilbert space. Note that Meng's proof is invalid in Banach space since it is based on the spectral theorem. Transfer to Banach space was obtained by us in 1987.

Theorem 6. *In smooth reflexive (or round reflexive) Banach space for the Lumeris norm-normal operators the followings are equivalent:*

1. *$V(T)$ is topologically closed*
2. *All extremal (or exposure) points of a convex shell of the spectrum $\sigma(T)$ lie in the point spectrum.*
3. *The sets specified in b) do not intersect with a continuous spectrum for T*

Definition. *We call a bounded linear operator T in any Banach space X a normally unitary if T is norm-normal and its spectrum lies on a unit circle. This is Meng's second result for a normally unitary operator.*

Theorem 7. *For the normally-unitary operator T in a reflexive smooth (or reflexive round) X numerical range $V(T)$ is topologically closed if the spectrum of T coincides with the set of its eigenvalues.*

For the criteria generalizing the previous authors see the list of our papers. For the general scheme of the proof of criteria including various classes of operators.

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Pattern formation in Ginzburg-Landau potentials with hard obstacles

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We present a new class of vector-valued phase field models, where the values of the phase parameter are constrained to a convex set. Like the classic Ginzburg-Landau functional, these models favour functions that partition the domain into subdomains, where the function takes one of a number of distinct values corresponding to distinct phases, separated by interfaces of small thickness. We characterise the phases and interfaces of the proposed generalized Ginzburg-Landau functional, in particular with respect to their dependency on the geometry of the convex constraint set. Furthermore, we introduce an efficient proximal gradient solver to study numerically their L^2-gradient flow, i.e. the associated generalized Allen-Cahn equation. We look at different choices for the shape of the convex constraint set, leading to the formation of a number of distinct patterns.

Optimality conditions for the control of the time-fractional wave equation

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Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and $T < \infty$. We consider a system whose state $y(x,t)$ is given by

$$ {}_0^C D_t^\alpha y(x,t) = Ay(x,t), \quad x \in \Omega, \quad t \in (0,T), \tag{1}$$

$$ y(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \tag{2}$$

$$ y(x,0) = 0, \quad x \in \Omega, \tag{3}$$

$$ \frac{\partial y}{\partial x}(x,0) = u(x), \quad x \in \Omega, \tag{4}$$

where ${}_0^C D_t^\alpha y(x,t)$ is left-sided Caputo fractional derivative of order $\alpha \in (1,2)$ with respect to t [1]:

$$ {}_0^C D_t^\alpha y(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-\tau)^{1-\alpha} \frac{\partial^2}{\partial \tau^2} y(x,\tau)\, \mathrm{d}\tau,$$

A is a symmetric uniformly elliptic operator on $\overline{\Omega}$:

$$ Ay(x) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial y}{\partial x_j}(x) \right) + a_0(x) y(x),$$

where $a_{ij} = a_{ji} \in C^1(\overline{\Omega})$, $a_0 \in C(\overline{\Omega}) \leq 0$ and there exists a constant $\mu > 0$ such that for each $x \in \overline{\Omega}$ and $\xi_1, \ldots, \xi_n \in \mathbb{R}$

$$ \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \geq \mu \sum_{i=1}^n \xi_i^2.$$

System (1)–(4) is used in particular in modeling of anomalous diffusion and mechanical wave propagation in viscoelastic media [2–3].

Let $U_\partial \subset H_0^1(\Omega)$ be a nonempty closed, convex set of admissible controls. We consider the following optimal control problem: find such a pair (y,u) that

$$ J(y,u) = \frac{1}{2} \left\| y(x,T) - y_d^0 \right\|_{L^2(\Omega)}^2 + \frac{1}{2} \left\| \frac{\partial y}{\partial x}(x,T) - y_d^1 \right\|_{L^2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{H_0^1(\Omega)}^2 \to \inf, \tag{5}$$

$$ u \in U_\partial, \tag{6}$$

where $y_d^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $y_d^1 \in D((-A)^\gamma) \subset L^2(\Omega)$, $\frac{1}{\alpha} \leq \gamma < 1$.

Theorem 1. *Problem* (1)–(6) *has a unique solution.*

We also consider an adjoint system

$$ {}_{t}^{RL}D_T^\alpha s(x,t) = As(x,t), \quad x \in \Omega, \quad t \in (0,T), \tag{7}$$

$$ s(x,t) = 0, \quad x \in \partial\Omega, \quad t \in (0,T), \tag{8}$$

$$ {}_0I_t^{2-\alpha} s(x,t)\big|_{t=T} = \frac{\partial y}{\partial x}(x,T) - y_d^1, \quad x \in \Omega, \tag{9}$$

$$ {}_0^{RL}D_t^{\alpha-1} s(x,t)\big|_{t=T} = y(x,T) - y_d^0, \quad x \in \Omega, \tag{10}$$

where ${}_0I_t^\alpha s(x,t)$ is the left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ with respect to t [1]:

$$ {}_0I_t^\alpha s(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} s(x,\tau)\,\mathrm{d}\tau,$$

and ${}_0^{RL}D_t^\alpha s(x,t)$ and ${}_t^{RL}D_T^\alpha s(x,t)$ are the left-sided and the right-sided Riemann-Liouville fractional derivatives of order $\alpha > 0$ with respect to t [1]:

$$ {}_0^{RL}D_t^\alpha s(x,t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\partial}{\partial t}\right)^n \int_0^t (t-\tau)^{n-\alpha-1} s(x,\tau)\,\mathrm{d}\tau,$$

$$ {}_t^{RL}D_T^\alpha s(x,t) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{\partial}{\partial t}\right)^n \int_t^T (\tau-t)^{n-\alpha-1} s(x,\tau)\,\mathrm{d}\tau,$$

where $n = [\alpha] + 1$.

Theorem 2. *If* (y,u) *is the optimal solution to* (1)–(6)*, then for all* $v \in U_\partial$

$$ \int_\Omega \left({}_0I_t^{2-\alpha} s(x,t)\right)\big|_{t=0} (v-u)\,\mathrm{d}x + \lambda\langle -\Delta u + u, v-u\rangle \geq 0,$$

where $s(x,t)$ *is the solution to* (7)–(10)*, and* $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$.

Conversely, if $y(x,t)$ *is the solution to* (1)–(4) *with* $u \in U_\partial$*,* $s(x,t)$ *is the solution to* (7)–(10)*, and for all* $v \in U_\partial$

$$ \int_\Omega \left({}_0I_t^{2-\alpha} s(x,t)\right)\big|_{t=0} (v-u)\,\mathrm{d}x + \lambda\langle -\Delta u + u, v-u\rangle \geq 0,$$

then (y,u) *is the optimal solution to* (1)–(6).

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On a nonlocal problem for a second-order fredholm integro-differential equation with a degenerate kernel

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In this paper, we study a nonlocal boundary value problem for an ordinary second-order Fredholm integro-differential equation with a degenerate kernel and two spectral parameters. The regular and irregular values of the spectral parameters are calculated, for which the solvability of the problem is established and the corresponding solutions are constructed in the case of their existence. A theory of solvability characteristic of this problem has been developed. The standard methods developed for studying the solvability of other types of differential and integro-differential equations do not work here.

Problem Statement. To find the function $u(t)$, satisfying on the interval $(0, T)$ the equation

$$u''(t) + \lambda^2 u(t) = \nu \int_0^T K(t,s)\ [s\, u(s) + (T-s)u'(s)]\ ds \tag{1}$$

and the following nonlocal conditions

$$u(T) - \int_0^T s\, u(s)\, ds = 0, \ \ u'(T) - \int_0^T (T-s)\, u'(s)\, ds = 0, \tag{2}$$

where $T > \sqrt{2}$, λ is positive parameter, ν is nonzero real parameter, $K(t,s) = \sum\limits_{i=1}^{k} a_i(t)\, b_i(s) \neq 0,\ a_i(t),\, b_i(s) \in C\,[0,T]$. We suppose that the functions $a_i(t)$ and $b_i(s)$ are linearly independent.

Since the boundary conditions (2) are homogeneous, the homogeneous integro-differential equation (1) always has trivial solutions. We investigate the existence of non-trivial solutions. We establish the uniqueness of the solution or build an infinite set of solutions. Let us determine for what values of the parameters λ and ν the problem has non-trivial solutions and construct these solutions.

Taking into account the degeneracy of the kernel, we write equation (1) in the following form

$$u''(t) + \lambda^2 u(t) = \nu \int_0^T \sum_{i=1}^{k} a_i(t)\, b_i(s)\ [s\, u(s) + (T-s)u'(s)]\ ds. \tag{3}$$

By the designation $\tau_i = \int_0^T b_i(s)\ [s\, u(s) + (T-s)u'(s)]\ ds$ equation (3) we rewrite it as

$u''(t)+\lambda^2 u(t) = \nu \sum\limits_{i=1}^{k} a_i(t)\,\tau_i$ and is solved by the method of variation of arbitrary constants

$$u(t) = A_1 \cos \lambda t + A_2 \sin \lambda t + \frac{\nu}{\lambda} \sum_{i=1}^{k} \tau_i \int_0^t \sin \lambda (t-s)\, a_i(s)\, ds, \tag{4}$$

where A_1, A_2 are arbitrary constants of integration. By differentiating (4), we obtain

$$u'(t) = -\lambda A_1 \sin \lambda t + \lambda A_2 \cos \lambda t + \nu \sum_{i=1}^{k} \tau_i \int_0^t \cos \lambda (t-s)\, a_i(s)\, ds. \tag{5}$$

To find the unknown coefficients A_1 and A_2 in (4), we use homogeneous integral conditions (2) and by virtue of (4) and (5), arrive at a system of linear algebraic equations

$$\begin{cases} A_1\chi_{11}(\lambda) + A_2\chi_{12}(\lambda) = \chi_{13}(\lambda), \\ A_1\chi_{21}(\lambda) + A_2\chi_{22}(\lambda) = \chi_{23}(\lambda), \end{cases} \tag{6}$$

where

$$\chi_{11}(\lambda) = \cos \lambda T - \frac{T}{\lambda} \sin \lambda T + \frac{1}{\lambda^2}(1 - \cos \lambda T),$$

$$\chi_{12}(\lambda) = \sin \lambda T + \frac{T}{\lambda} \cos \lambda T - \frac{1}{\lambda^2} \sin \lambda T,$$

$$\chi_{21}(\lambda) = -\lambda \sin \lambda T - T - \frac{1}{\lambda} \sin \lambda T, \;\; \chi_{22}(\lambda) = \lambda \cos \lambda T - \frac{1}{\lambda}(1 - \cos \lambda T),$$

$$\chi_{13}(\lambda) = -\eta(T,\lambda) + \int_0^T s \cdot \eta(s,\lambda)\, ds, \;\; \chi_{23}(\lambda) = -\eta'(T,\lambda) + \int_0^T (T-s) \cdot \eta'(s,\lambda)\, ds,$$

$$\eta(t,\lambda) = \frac{\nu}{\lambda} \sum_{i=1}^{k} \tau_i h_i(t,\lambda), \;\; h_i(t,\lambda) = \int_0^t \sin \lambda (t-s)\, a_i(s)\, ds, \;\; i = \overline{1,k}.$$

We will analyze the system of linear algebraic equations (6) with respect to parameter λ. We find that in the following two cases
1) $\chi_{11}(\lambda) = \chi_{12}(\lambda) = 0$, 2) $\chi_{21}(\lambda) = \chi_{22}(\lambda) = 0$ one should additionally check the correctness of the formulated problem (1), (2). If the problem is correctly posed, then construct all solutions to this problem. If the problem is not set correctly, then there are only trivial solutions.

We find that in the following three cases: 3) $\chi_{11}(\lambda) = \chi_{21}(\lambda) = 0$, 4) $\chi_{12}(\lambda) = \chi_{22}(\lambda) = 0$, 5) $\chi_{11}(\lambda)\,\chi_{22}(\lambda) - \chi_{12}(\lambda)\chi_{21}(\lambda) = 0$ the uniqueness of the solution of the stated problem (1), (2) is violated. In these cases, we find sufficient conditions for the existence of solutions and construct these solutions.

We find that in the following three cases: 6) $\chi_{11}(\lambda) = \chi_{21}(\lambda) \neq 0$, 7) $\chi_{12}(\lambda) = \chi_{22}(\lambda) \neq 0$, 8) $\chi_{11}(\lambda)\,\chi_{22}(\lambda) - \chi_{12}(\lambda)\chi_{21}(\lambda) \neq 0$ the uniqueness of the solution of the stated problem

(1), (2) is not violated. In these cases, we find sufficient conditions for the existence of a unique solution and construct this solution.

In this paper, we also study other cases related to regular and irregular values of the second parameter ν. This work differs from work [1] not only in the method of research, but also in the content of the obtained results.

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Classification according to NDVI (normalized difference vegetation index) values, evaluation of discriminating threshold value in soil-vegetation classification

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When studying the spectrum of soil and vegetation cover, various empirical indicators are calculated. These empirical indicators are calculated using mathematical operations performed on the basis of spectral images for various wavelengths of soil and vegetation. An important difficulty in using spectral images in many practical problems is the choice of a threshold brightness value. It is known from the scientific literature that observing the change in NIR (infrared light) compared to RED (red light) allows you to determine the presence of chlorophyll associated with plant health. In the known literature, an NDVI discriminant value of 0.2 is mainly used to distinguish between vegetative and non-vegetative parts based on NDVI: pixels with an NDVI value greater than 0.2 are considered vegetation, and pixels with an NDVI value less than 0.2 are considered non-vegetable pixels. It should be noted that in fact this separating value is not a specific number, but can be taken as a certain range of numbers. In the present work, this range of separating numbers was selected and a comparative analysis of the threshold value was carried out.

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Mathematical methods in technical designs

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Many properties of technical solutions can be described by a selection of appropriate technical parameters that characterize them. Their authors, called inventors, receive a form of protection for their solutions, called a patent. Studying them it is easy to see that they are not always characterized by the extreme value of parameters in relation to the utility function. This element is often overlooked in the patenting process. It also happens that it is very difficult to choose the right tools and mathematical methods that would allow solving such a problem.

In the presented lecture, several technical solutions as patents will be shown, with extreme features of the parameters, which were obtained with the help of both new and known mathematical methods.

On solvability of homogeneous Riemann boundary value problems in Hardy-Orlicz classes

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This work deals with the Orlicz space and the Hardy-Orlicz classes generated by this space, which consist of analytic functions inside and outside the unit disk. The homogeneous Riemann boundary value problems with piecewise continuous coefficients are considered in these classes.

We will use the following notations. R will stand for the set of real numbers, by C we will denote the set of complex numbers, $\omega = \{z \in C : |z| < 1\}$ will denote a unit disk in C, $\partial\omega$ will be a unit circle.

Definition 1. *Continuous convex function $M(u)$ in R is called an N-function if it is even and satisfies the conditions*

$$\lim_{u\to 0}\frac{M(u)}{u} = 0; \ \lim_{u\to\infty}\frac{M(u)}{u} = \infty.$$

Definition 2. *Let M be an N-function. The function*

$$M^*(v) = \max_{u\geq 0}\left[u\,|v| - M(u)\right],$$

is called an N-function complementary to $M(\cdot)$.

The functions

$$M(u) = \int_0^{|u|} p(t)\,dt; M^*(v) = \int_0^{|v|} q(s)\,ds,$$

are called N-functions complementary to each other.

Definition 3. *N-function $M(\cdot)$ satisfies Δ_2-condition for large values of u, if $\exists k > 0 \wedge \exists u_0 \geq 0$:*

$$M(2u) \leq kM(u), \forall u \geq u_0.$$

The set of N-functions satisfying Δ_2-condition will be denoted by $\Delta_2(\infty)$.

Let $M(\cdot)$ be some $N-$function. As usual, by H_M^+ we denote the Hardy-Orlicz class of analytic functions $F(\cdot)$ inside ω equipped with the norm

$$\|F\|_{H_M^+} = \sup_{0<r<1}\ \sup_{\rho_{M^*}(\nu)\leq 1}\left|\left(F_r(\cdot);\nu(\cdot)\right)\right| = \sup_{0<r<1}\|F_r(\cdot)\|_M,$$

where $F_r(t) = F\left(re^{it}\right)$.

Similar to the classical case, we define the Hardy-Orlicz class ${}_mH_M^-$ of analytic functions outside the unit disk which have a finite order at infinity.

Consider the homogeneous Riemann problem

$$\begin{array}{l} F^+(\tau) - G(\tau)F^-(\tau) = 0, \tau \in \gamma, \\ F^+(\cdot) \in H_M^+; F^-(\cdot) \in {}_mH_M^-, \end{array} \tag{1}$$

with a complex-valued constant $G\left(e^{it}\right) \equiv \left|G\left(e^{it}\right)\right| e^{i\theta(t)}, t \in [-\pi, \pi]$. By the solution of the problem (1) we mean a pair of analytic functions $(F^+; F^-) \in H_M^+ \times {}_mH_M^-$, whose non-tangential boundary values satisfy the equation (1) a.e. on $\partial\omega$.

Consider the following piecewise continuous functions on the complex plane with a cut $\partial\omega$:

$$\begin{array}{l} Z_1(z) \equiv \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \log\left|G\left(e^{it}\right)\right| \frac{e^{it}+z}{e^{it}-z} dt\right\}, \\ Z_2(z) \equiv \exp\left\{\frac{i}{4\pi}\int_{-\pi}^{\pi} \theta(t) \frac{e^{it}+z}{e^{it}-z} dt\right\}, z \notin \partial\omega. \end{array}$$

Let

$$Z_\theta(z) = Z_1(z) Z_2(z), z \notin \partial\omega.$$

We will call $Z_\theta(\cdot)$ a canonical solution of homogeneous problem (1), corresponding to the argument $\theta(\cdot)$.

We will assume that the coefficient $G(\cdot)$ satisfies the following conditions:

i) $G^{\pm 1}(\cdot) \in L_\infty(-\pi, \pi)$;

ii) $\theta(t) = \arg G\left(e^{it}\right)$ is a piecewise Hölder function on $[-\pi, \pi]$ with the jumps $h_k = \theta(s_k+0) - \theta(s_k-0), k = \overline{1,r}$, at the points of discontinuity $\{s_k\}_1^r : -\pi < s_1 < \ldots < s_r < \pi$.

The following theorem is proved.

Theorem 1. Let $M \in \Delta_2(\infty)$ be some N-function, and $M^*(\cdot)$ be an N-function complementary to M. Suppose the coefficient $G(\cdot)$ of the problem (1) satisfies the conditions *i*), *ii*) and $Z_\theta(\cdot)$ is a canonical solution corresponding to the argument $\theta(\cdot)$. Let the jumps $\{h_k\}_0^r$ of the function $\theta(\cdot)$, where $h_0 = \theta(-\pi) - \theta(\pi)$, satisfy the inequalities

$$\gamma_{M^*} < \frac{h_k}{2\pi} < -\gamma_M, \ k = \overline{0,r}.$$

Then:

α) for $m \geq 0$ the homogeneous Riemann problem (1) has a general solution of the form

$$F(z) = Z_\theta(z) P_k(z),$$

in the Hardy-Orlicz classes $H_M^+ \times_m H_M^-$, where $P_k(\cdot)$ is an arbitrary polynomial of degree $k \leq m$; β) for $m < 0$ this problem has only a trivial, i.e. zero solution.

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Obtaining necessary conditions for the local boundary condition problem for a first order elliptic equation

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In this work, the problem is considered within the local boundary condition, which contains the real and imaginary parts of the analytic function sought for the Cauchy-Riemann equation. It should be noted that the investigation of the solution is carried out by means of the obtained necessary conditions [1].

Let us consider the following boundary value problem:

$$\frac{\partial u(x)}{\partial x_2}+i\frac{\partial u(x)}{\partial x_1}=0, \quad x=(x_1,x_2)\,\epsilon D\subset R^2, \tag{1}$$

$$\alpha_1(x)\,u_1(x)+\alpha_2(x)\,u_2(x)=\alpha_0(x)\,, \quad x=(x_1,x_2)\,\epsilon\partial D \tag{2}$$

where $i=\sqrt{-1}$, D is a bounded smooth domain, Γ - boundary of the domain is a Lyapunov line, $\alpha_k(x)\quad k=\overline{0,2;}\ \ x\epsilon\Gamma$ are continous functions, u$(x)=u_1(x)+iu_2(x)\,.$

It is known that the fundamental solution of equation (1) is of the form [2]:

$$U(x-\xi)=\frac{1}{2\pi}\cdot\frac{1}{x_2-\xi_2+i(x_1-\xi_1)} \tag{3}$$

Multiplying the equation (1) by the fundamental solution (3), integrating with respect to the domain D, applying the Ostrogradsky-Gauss formula, i.e. integration by parts, we have [2]:

$$0=\int_D\frac{\partial u(x)}{\partial x_2}U(x-\xi)\,dx+i\int_D\frac{\partial u(x)}{\partial x_1}U(x-\xi)\,dx=$$
$$=\int_\Gamma(x)\,U(x-\xi)\,[cos(\nu,x_2)+icos(\nu,x_1)]\,dx-$$
$$-\int_D u(x)\left[\frac{\partial U(x-\xi)}{\partial x_2}+i\frac{\partial U(x-\xi)}{\partial x_1}\right]dx_1$$

Here, if we consider (3) and use the property of the delta function, we get:

$$\frac{1}{2\pi}\int_\Gamma\frac{u(x)}{x_2-\xi_2+i(x_1-\xi_1)}\,[cos(\nu,x_2)\ \ +icos(\nu,x_1)]\,dx=$$
$$=\begin{cases} u(\xi)\,,\xi\epsilon D,\\ \frac{1}{2}u(\xi)\,,\ \xi\epsilon\Gamma,\end{cases} \tag{4}$$

so that, ν is the external normal drawn to the boundary Γ of the domain D. The second expression of the main relation we obtain (4) is a necessary condition. Let us separate this.

$$\frac{1}{2}\left(u_1\left(\xi\right)+iu_2(\xi)\right)=\frac{1}{2\pi}\int_{\Gamma}\left[u_1\left(x\right)+iu_2(x)\right]\frac{x_2-\xi_2-i(x_1-\xi_1)}{\left|x-\xi\right|^2}\times$$

$$\times\left[\cos\left(\nu,x_2\right)\ +icos\left(\nu,x_1\right)\right]dx,\xi\epsilon\Gamma,$$

$$\frac{1}{2}u_1(\xi)=\frac{1}{2\pi}\int_{\Gamma}\frac{\left[(x_2-\xi_2)cos(\nu,x_2)+(x_1-\xi_1)cos(\nu,x_1)\ \right]u_1(x)-}{\left|x-\xi\right|^2}$$

$$\frac{-\left[(x_2-\xi_2)\,cos\left(\nu,x_1\right)\ -(x_1-\xi_1)\,cos\left(\nu,x_2\right)\ \right]u_2\left(x\right)}{\left|x-\xi\right|^2}dx,$$

$$\frac{1}{2}u_2(\xi)=\frac{1}{2\pi}\int_{\Gamma}\frac{\left[(x_2-\xi_2)cos(\nu,x_2)\ +(x_1-\xi_1)cos(\nu,x_1)\ \right]u_2(x)-}{\left|x-\xi\right|^2}$$

$$\frac{-\left[(x_2-\xi_2)\cos\left(\nu,x_1\right)\ -(x_1-\xi_1)\,cos\left(\nu,x_2\right)\ \right]u_1\left(x\right)}{\left|x-\xi\right|^2}dx,\xi\varepsilon\Gamma, \tag{5}$$

we get (5) necessary conditions which exist in the singular integrals.

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